Disjointness preservers of operator algebras

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I will report my 25 year struggle with the problem of whether a linear disjointness preserver T between two C*-algebras A and B is close to an algebra/Jordan (*-)homomorphism. Here, the disjointness of two elements a, b in a C*-algebra can have at least 5 versions; namely,

$$ab = 0, a^*b = 0, ab^* = 0, a^*b = ab^* = 0, \text{ or } ab = ba = 0.$$

When a, b are self-adjoint, all reduce to the zero product case $ab = 0$. For example, we ask what happens if T preserves zero products.

We attack the problem in several steps.

It is pretty fruitful when A, B are commutative C*-algebras. It is also satisfactory when A, B are matrix algebras. In both cases, a linear zero product preserver $T : A \to B$ is merely a direct sum of a good part hJ and a 'bad but small' part. Here, h is a central element affiliated with B, and J is a Jordan homomorphism. The bad part arises either from the discontinuity, or the non-surjectivity of T which translates into that the range TA is not a C*-algebra.

In general, we need to assume either the continuity or the surjectivity of T. When T is a continuous linear disjointness preserver, we have a quite complete answer. It merely says that $T = hJ$ is almost good.

For a surjective linear disjointness preserver $T : A \to B$, we have good answers for the cases when A, B are W*-algebras or AW*-algebras. The solutions are obtained by decomposing T as a direct sum of surjective linear disjointness preservers between the abelian summands and between the orthogonal complements. While the abelian part involves mainly topological arguments, the nonabelian part relies on techniques dealing with projections.

The major obstruction to carry on to the C*-algebra case is due to the lack of nontrivial projections for general C*-algebras. We managed to finish the cases when A, B are type I or properly infinite. For a complete answer, we devote a great efforts to the developing of two type decomposition theories for general C*-algebras, so that any C*-algebra A has an essential ideal which is a direct sum of type A finite, type A infinite, type B finite, type B infinite, and type C. Our plan is to solve the problem for surjective linear disjointness preservers between C*-algebras of each of these 5 types,

and 'glue' the 5 versions together for a complete answer. Unfortunately, we have encountered big difficulties and struck for some years.

We have also worked on other preserver problems for motivations, e.g. linear orthogonality preservers of Hilbert C*-modules, and orthogonal additive conformal disjointness preservers of C*-algebras.

In this talk, I will present some details of the above experience. The above works are contributions of many authors. Some of them are in the audience, for example, Chi-Kwong Li, Lajos Molnar and Antonio Peralta, as well as some of my long time collaborators, Chi-Wai Leung, Lei Li, Chi-Keung Ng, Ming-Cheng Tsai and Ya-Shu Wang, to name a few.

I hope the audience can help to solve the problem.

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