Chromatic numbers and conflict free chromatic numbers of (strongly)almost disjoint set-systems

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Basic definition

A set $\mathcal{A}$ is a **set-system** if all elements of it have size at least 2.

f is a **proper coloring** of a set-system $\mathcal{A}$ iff $|f''A| \geq 2$ for each $A \in \mathcal{A}$.

f is a **conflict free coloring** of a set-system $\mathcal{A}$ with ρ colors iff $f : \cup\mathcal{A} \to \rho$ and $\forall A \in \mathcal{A}$ $(\exists \zeta < \rho)$ $|A \cap f^{-1}\{\zeta\}| = 1$.

f is a **weak conflict free coloring** iff we weaken the assumption $\mathrm{dom}(f) = \cup\mathcal{A}$ to $\mathrm{dom}(f) \subset \cup\mathcal{A}$.

The **chromatic number**, the **conflict-free chromatic number** and the **weak conflict-free chromatic number** of a set-system $\mathcal{A}$, denoted by

$$\chi(\mathcal{A}),\ \chi_{\mathrm{CF}}(\mathcal{A}),\ \mathrm{w}\chi_{\mathrm{CF}}(\mathcal{A})$$

respectively, are defined as the **minimal number of colors** in a **proper**, **conflict free** or a **weak conflict-free coloring**, respectively.

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- $\chi(\mathcal{A}) = \chi_{\mathrm{CF}}(\mathcal{A})$ provided $|A| \leq 3$ for all $A \in \mathcal{A}$
- For all $\kappa \geq \omega$ there exists a quadruple system $\mathcal{A}$ with $\chi(\mathcal{A}) = 2$ and $\chi_{\mathrm{CF}}(\mathcal{A}) = \kappa$:
  $\mathcal{A} = \{H \in [\kappa]^4 :\ H$ contains even and odd ordinals as well$\}$.

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- **Part 3. More ZFC results**

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- **Part 3. More ZFC results**

Outline

- $f : \cup\mathcal{A} \to \rho$ is a **CF-coloring** iff $\forall A \in \mathcal{A}\ (\exists\zeta < \rho)\ |A \cap f^{-1}\{\zeta\}| = 1$.
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$\mathcal{A}$ is d-almost disjoint for some $d < \omega$.

- $[\lambda, \kappa, \mu] \to \rho$ iff every μ-almost $\mathcal{A} \subset [\lambda]^{\kappa}$ has a CF-colorings with ρ-colors
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For each infinite cardinals $\kappa \leq \lambda$ and $d \in \omega$ we have $[\lambda, \kappa, d] \to \omega$.

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Proof: $[\lambda, \kappa, d] \to \omega$

$I_c(A) = \{\xi : |c^{-1}\{\xi\} \cap A| = 1\}$ c **is CF iff** $I_c(A) \neq \emptyset$

$(\star_{\kappa,\lambda})$ $\forall$ d-AD family $\mathcal{A} = \{A_\nu : \nu < \lambda\} \subset [\lambda]^\kappa$ there is $f : \lambda \to \omega$ s.t.
$(\forall \nu < \lambda)$ $c''A_\nu = \omega$ and $\omega \setminus I_c(A_\nu)$ **is finite**.

Case 3: $\lambda > \kappa$.

- Let $\langle N_\alpha : \alpha < \lambda \rangle$ be a λ-**chain of models** with $(\kappa + 1) \cup \{\mathcal{A}\} \subset N_0$.
- **If** $A \in \mathcal{A} \setminus N_\alpha$ **then** $|A \cap N_\alpha| < d$
- $\mathcal{A}'_\alpha = \{A \setminus N_\alpha : A \in N_{\alpha+1} \setminus N_\alpha\} \subset [(N_{\alpha+1} \setminus N_\alpha) \cap \lambda]^\kappa$ is d-AD.
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Case 1: $\lambda = \kappa = \omega$. Any injective function $c : \omega \to \omega$ works.

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Case 2: $\lambda = \kappa > \omega$.

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Case 3: $\lambda > \kappa$.

- Let $\langle N_\alpha : \alpha < \lambda \rangle$ be a **λ-chain of models** with $(\kappa + 1) \cup \{\mathcal{A}\} \subset N_0$.
- **If $A \in \mathcal{A} \setminus N_\alpha$ then $|A \cap N_\alpha| < d$**
- $\mathcal{A}'_\alpha = \{A \setminus N_\alpha : A \in N_{\alpha+1} \setminus N_\alpha\} \subset [(N_{\alpha+1} \setminus N_\alpha) \cap \lambda]^\kappa$ is d-AD.
- $\exists c_\alpha : (N_{\alpha+1} \setminus N_\alpha) \cap \lambda \longrightarrow \omega$ s.t. $c''_\alpha A' = \omega$ and $\omega \setminus I_{c_\alpha}(A')$ **is finite** for all $A' \in \mathcal{A}'_\alpha$.
- **$c = \cup\{c_\alpha : \alpha < \kappa\}$ witnesses $(\star_{\kappa,\lambda})$.**
- Indeed, let $A \in \mathcal{A} \cap (N_{\alpha+1} \setminus N_\alpha)$. Then $A' = A \cap (N_{\alpha+1} \setminus N_\alpha) \in \mathcal{A}'_\alpha$. $c''A \supset c''_\alpha A' = \omega$. Moreover $|\mathrm{dom}(c \setminus c_\alpha) \cap A| < d$.
- Thus $\omega \setminus I_c(A) \subset (\omega \setminus I_{c_\alpha}(A')) \cup (I_{c_\alpha}(A') \setminus I_c(A))$, so $\omega \setminus I_c(A)$ is finite.

Proof: $[\lambda, \kappa, d] \to \omega$

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- **Do we really need ω colors?**
- **Special case**: consider the 2-dimensional vector space V above $\mathbb{Q}$, and let $\mathcal{E}$ be the family of the lines of V.
- $\chi_{\mathrm{CF}}(\mathcal{E}) = 3$,
- if $\mathcal{A} \subset [\omega]^{\omega}$ is 2-almost-disjoint then $\chi_{\mathrm{CF}}(\mathcal{A}) \leq 3$.
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$\mathcal{A}$ is ω-almost disjoint

ω_2 colors

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$[\lambda, \kappa, \omega] \to \omega_2$

- $[\lambda, \kappa, d] \to \omega$: If $\mathcal{A} \subset [\lambda]^{\kappa}$ is d-almost disjoint, $\mathcal{A} \in N \prec \mathcal{H}(\theta)$, $A \in \mathcal{A} \setminus N$, then $|A \cap N| < d$.
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$\langle M_\alpha : \alpha < \mu^+ \rangle \prec \mathcal{H}(\vartheta)$ is a a **(ω_1, μ)-dominating sequence over x** iff

(s1) $(\omega_1 + 1) \cup \{x\} \subset M_0$, $|M_\alpha| \le \mu$ and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$,

(s2) for each $\alpha < \mu^+$ the set M_α is the union of sets $\{M'_{\alpha,n} : n < \omega\}$ such that $\left[M'_{\alpha,n}\right]^\omega \subset M_\alpha$.

Theorem

Let λ be an infinite cardinal. Assume that

(i) $\mu^\omega = \mu^+$ *for each cardinal* $\mu < \lambda$ *with* $\mathrm{cf}(\mu) = \omega$,

(ii) *for each cardinal* $\mu < \lambda$ *with* $\mathrm{cf}(\mu) = \omega$ *if* ϑ *is sufficiently large and* $x \in \mathcal{H}(\vartheta)$ *then there is a* (ω_1, μ)*-dominating sequence over* x.

Then $[\lambda, \kappa, \omega] \to \omega_1$ *for each* $\omega_1 \le \kappa \le \lambda$.

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Theorem (Komjáth)

There is an ω-almost disjoint family $\mathcal{A} \subset [2^\omega]^\omega$ with $\chi(\mathcal{A}) = 2^\omega$. Hence $\chi_{\mathrm{CF}}([2^\omega, \omega, \omega]) = 2^\omega$.

Komjáth proved that there is an ω-almost disjoint family $\mathcal{A} \subset [2^\omega]^\omega$ such that for each $X \in [2^\omega]^{\omega_1}$ there is $A \in \mathcal{A}$ with $A \subset X$.

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If CH holds then $[\omega_1, \omega_1, \omega] \not\to \omega$ *and* $[\omega_1, \omega, \omega] \not\to \omega$.

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More ZFC result

- Closure operation: Find $\langle N_\alpha : \alpha < \kappa \rangle$ s.t **closed enough**
- $\rho^{[\nu]} = \rho$ iff there is a family $\mathcal{B} \subset [\rho]^{\leq \nu}$ of size ρ such that for all $u \in [\rho]^{\nu}$ there is $\mathcal{P} \in [\mathcal{B}]^{<\nu}$ such that $u \subset \cup \mathcal{P}$.
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On set-systems of finite sets.

Theorem

Let k and n be natural numbers, $k \geq 2$. Write $t = (n+1)(k-1)$.
(1) Every k-almost disjoint family $\mathcal{A} \subset [\omega_n]^{t+1}$ has a conflict free coloring with ω colors.
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- $[\omega_2, 4, 2] \rightarrow \omega$, and $2^\omega = \omega_1$ implies $[\omega_2, 3, 2] \not\rightarrow \omega$,
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On set-systems of finite sets.

Theorem

Let k and n be natural numbers, $k \geq 2$. Write $t = (n+1)(k-1)$.
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