Variations of Separability

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Winter School in Abstract Analysis
section Set Theory

Outline

properties stating that a space has a small dense set

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Evolution

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- Suslin Problem

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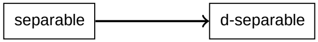
- Kurepa: property K_0

Evolution



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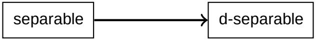

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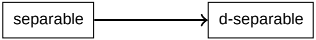

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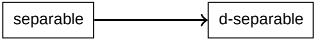

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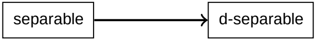

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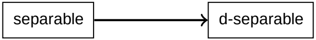

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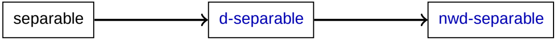

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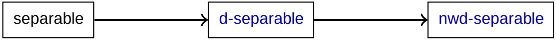

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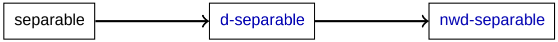

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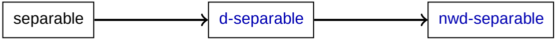

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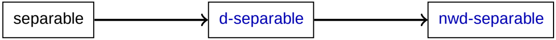

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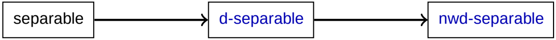

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- there is a compact nwd-separable space which is not d-separable: $X = \omega^* \times D(2)^\omega$

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separable $\longrightarrow$ d-separable $\longrightarrow$ nwd-separable

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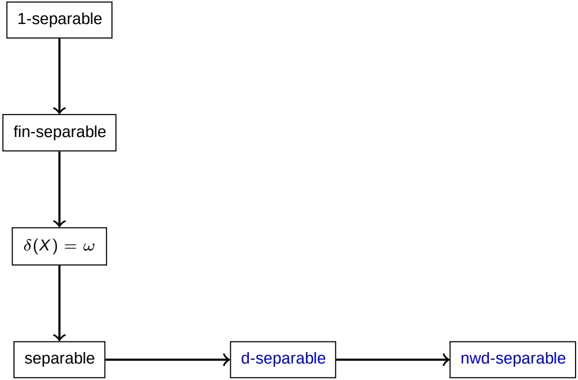
- fin-separable $\Longrightarrow \delta(X) = \omega$ (every dense subset is separable)

Selection Principles



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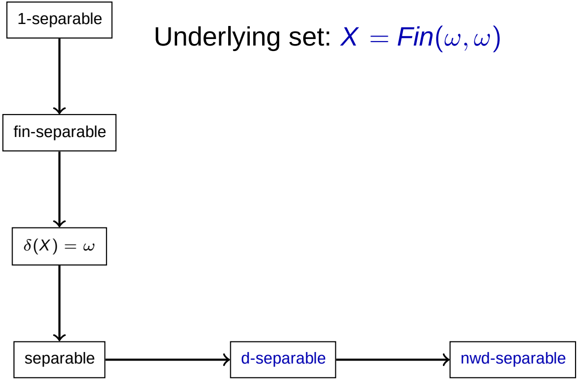

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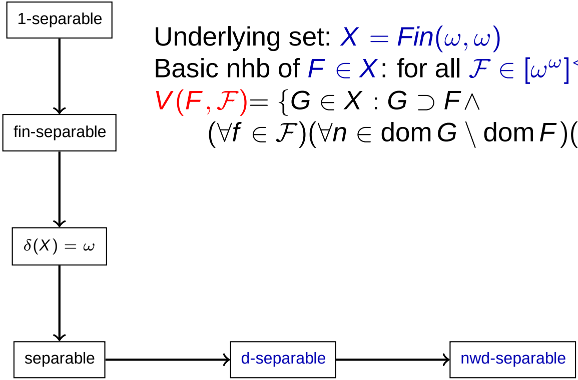

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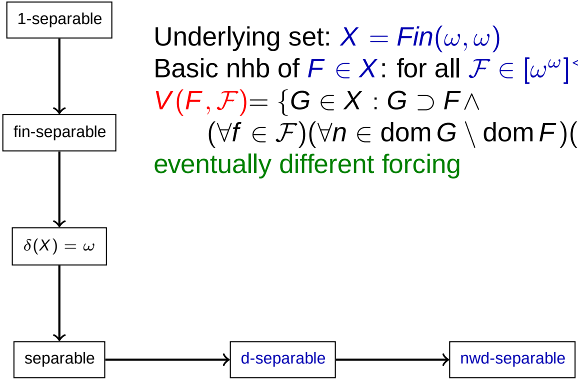
eventually different forcing



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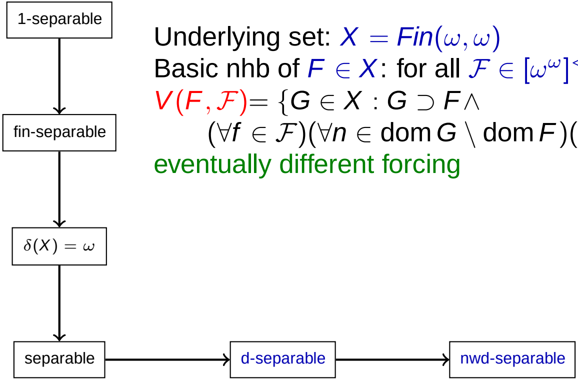

- How to prove that X is 1-separable?

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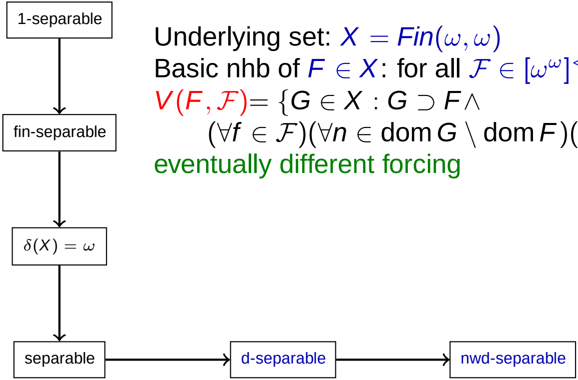

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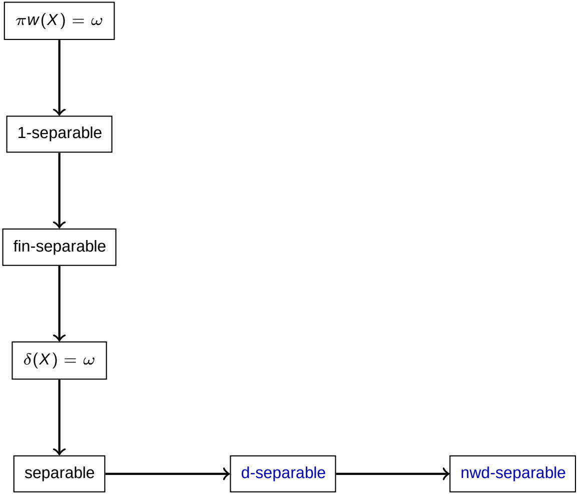
- How to prove that X is 1-separable?
- $\pi(X) = \omega$

Classical positive results



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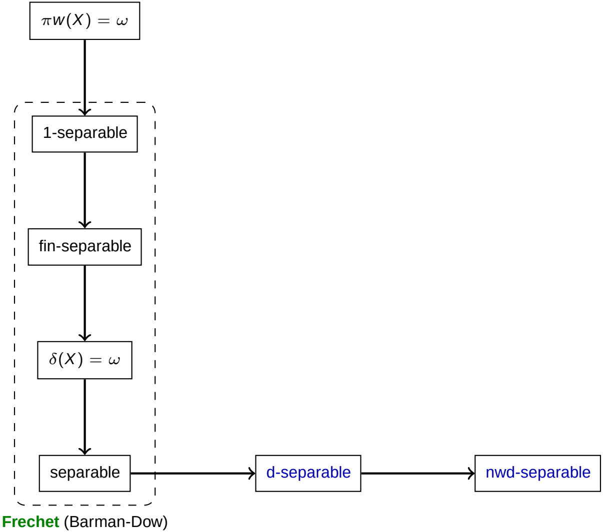

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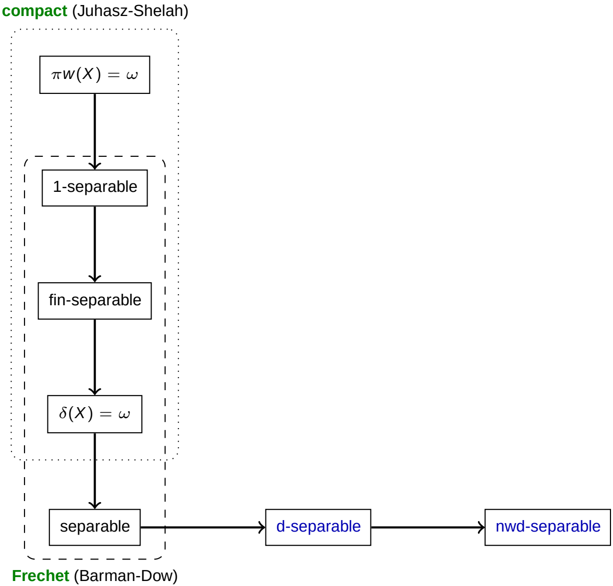

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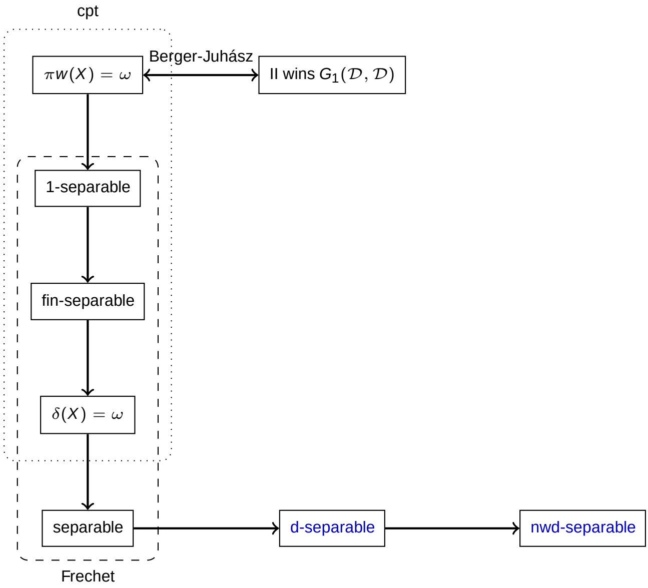

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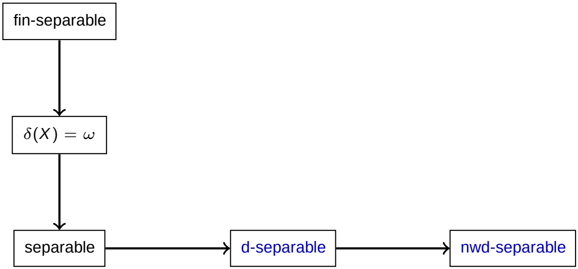

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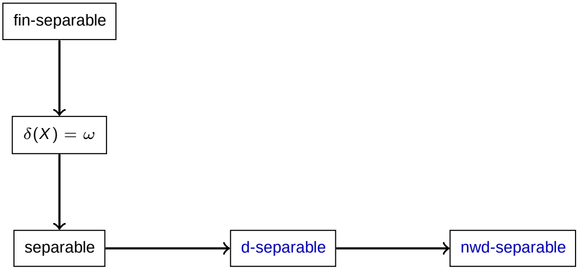

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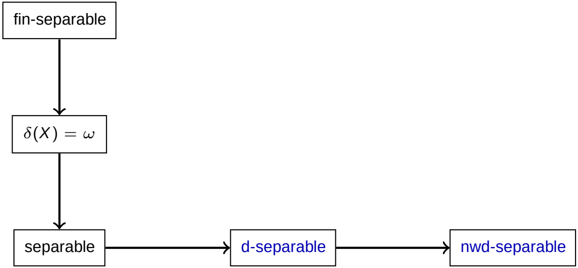

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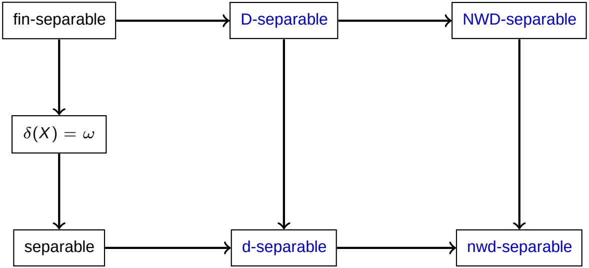

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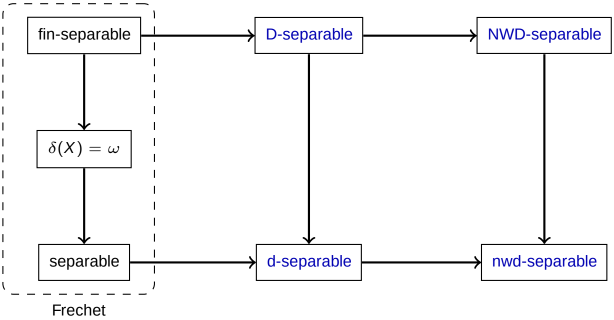
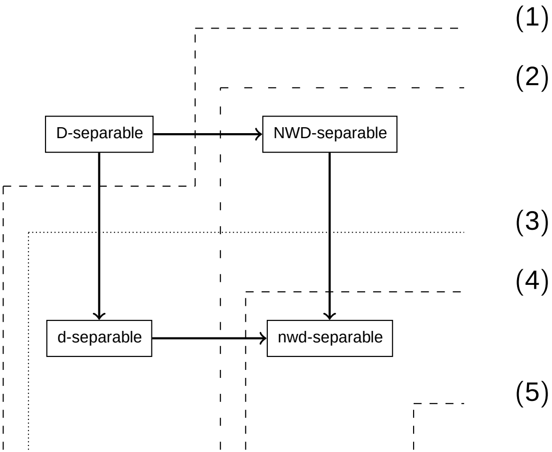


A separation theorem



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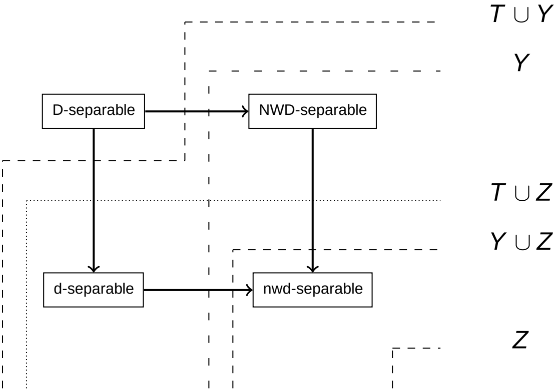

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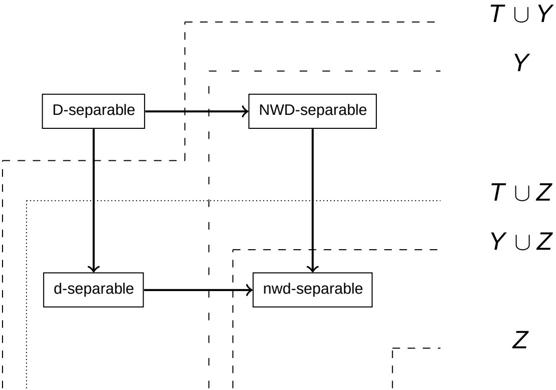

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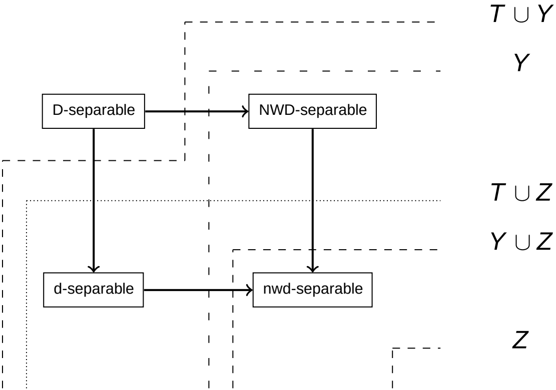

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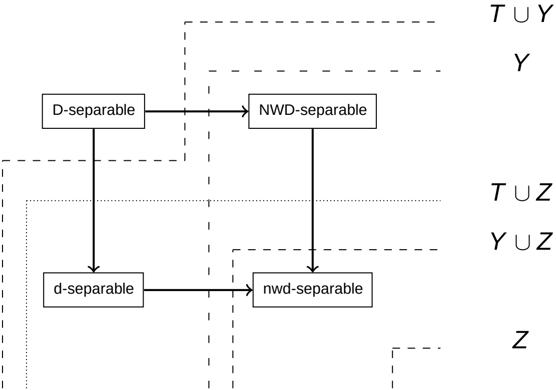

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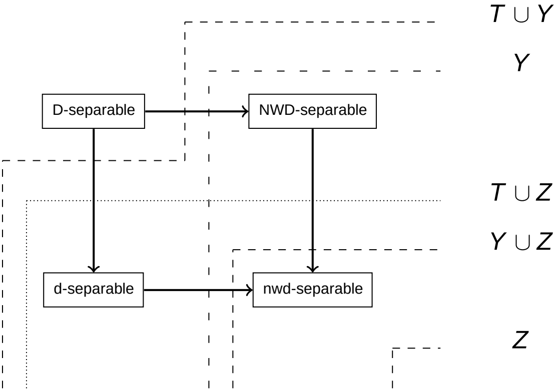

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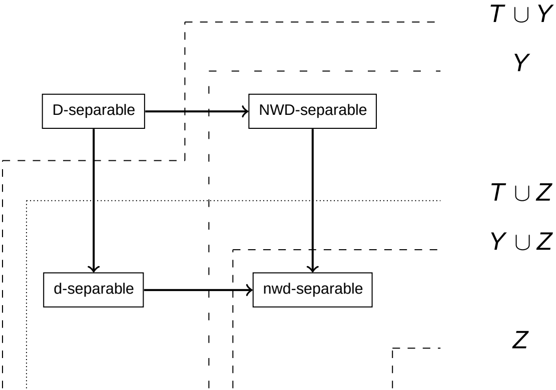

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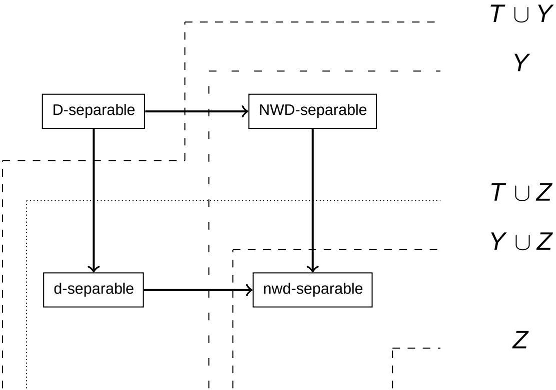
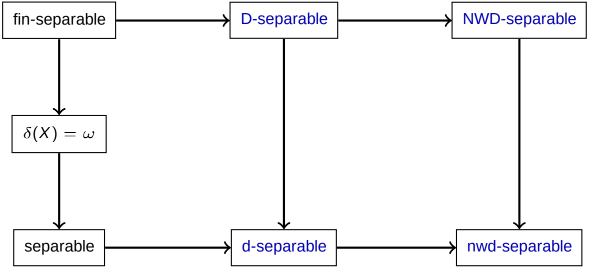


Selection Principles



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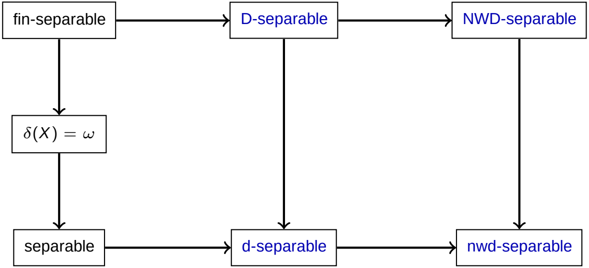

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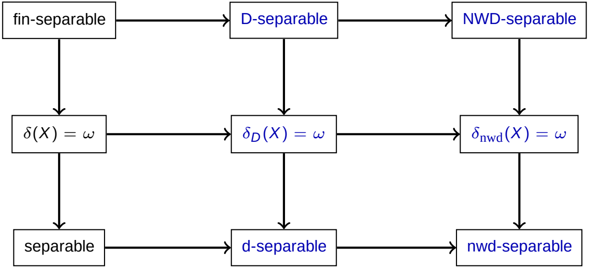

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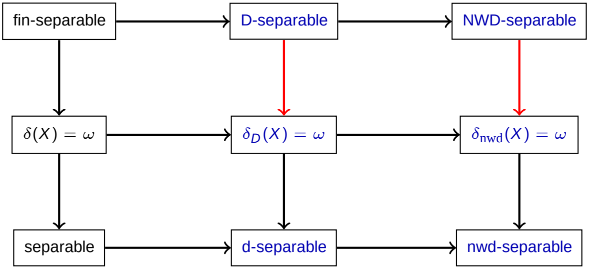

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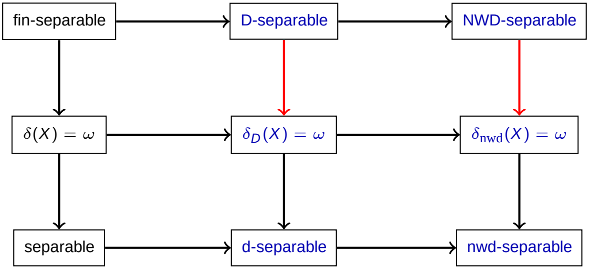

- ZFC examples?
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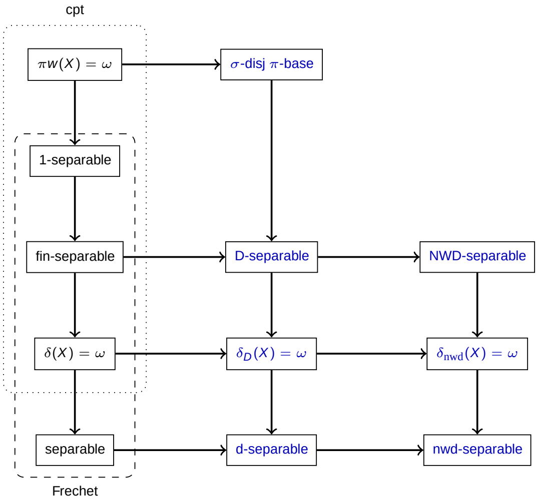
Positive theorems



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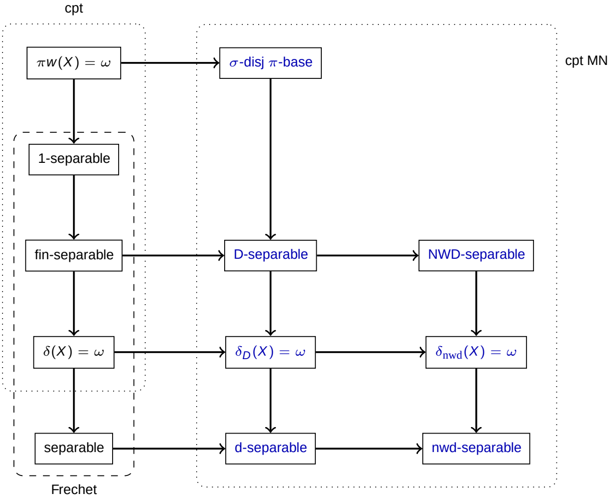

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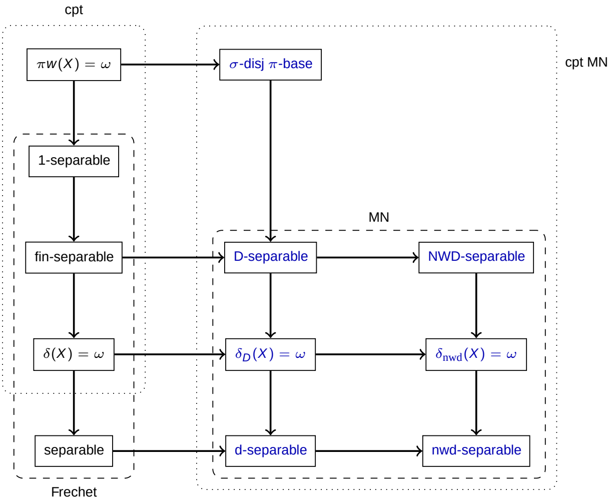

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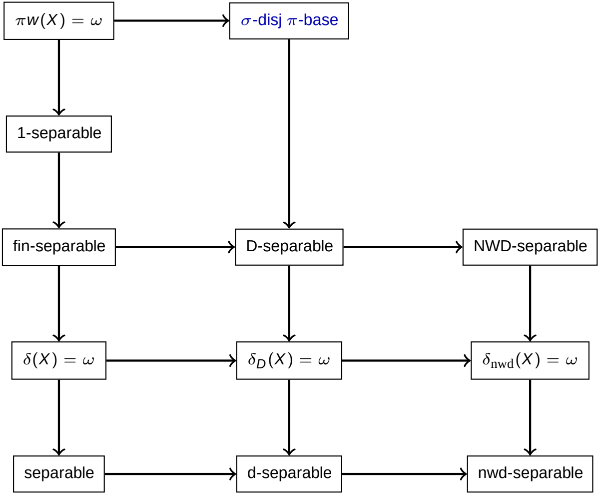
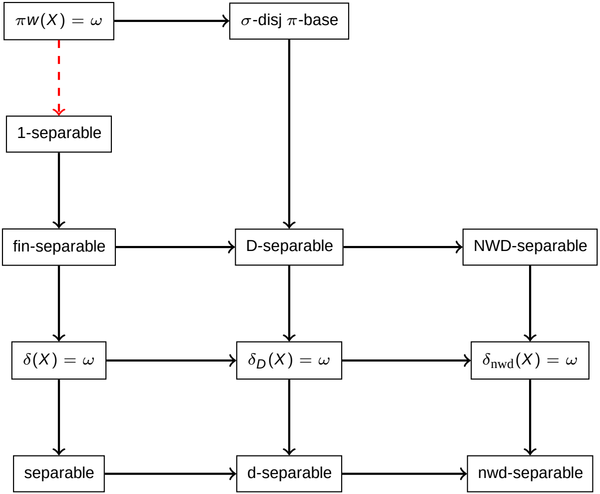
- MN nwd-separable spaces are D-separable

ZFC results

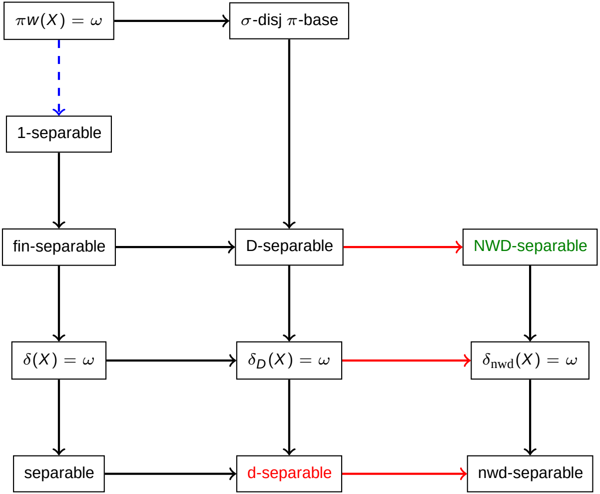

$\pi w(X) = \omega$
σ-disj π-base
1-separable
fin-separable
D-separable
NWD-separable
$\delta(X) = \omega$
$\delta_D(X) = \omega$
$\delta_{\mathrm{nwd}}(X) = \omega$
separable
d-separable
nwd-separable

ZFC results



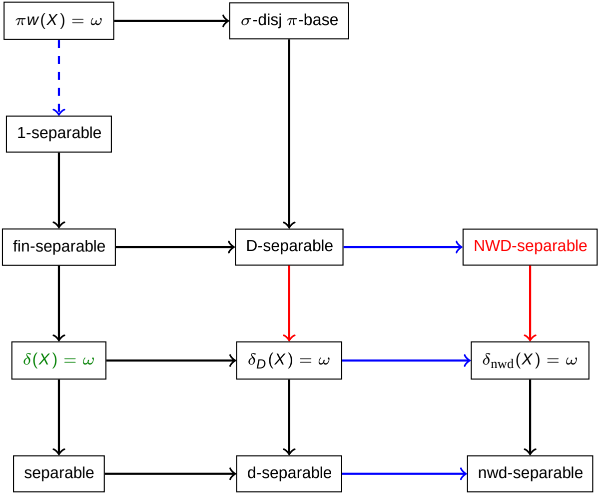
- Only consistent example is known: a countable HFC.

ZFC results



- If $Y = X \times \mathbb{Q}$, where X is the G_δ topology on $D(2)^{\omega_1}$, then Y is NWD-separable, but not d-separable.

ZFC results



- There is a countable dense subspace of $D(2)^{\mathfrak{c}}$ which is not NWD-separable.

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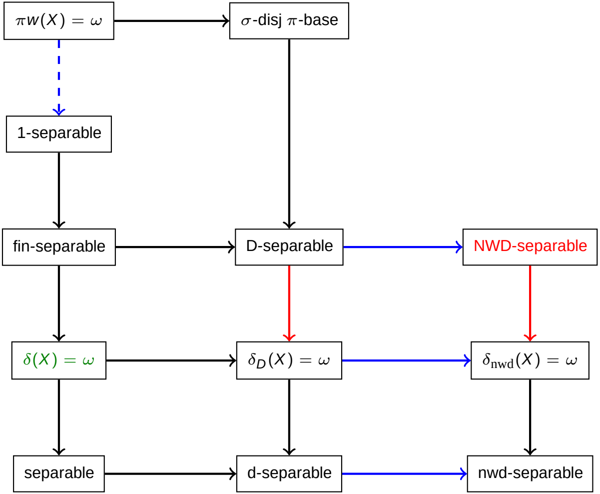
- Then X is not NWD-separable.
- If $E_n \subset D_n$ is nowhere dense, then $E = \bigcup_{n\in\omega} E_n$ is not dense, because it can not contain any $D_n \cap U$.

ZFC results



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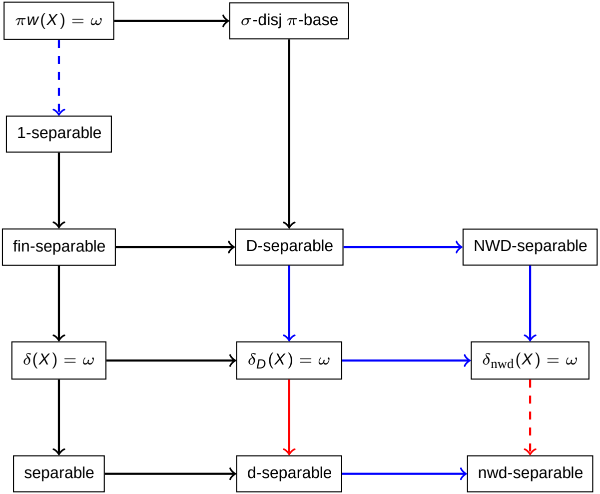

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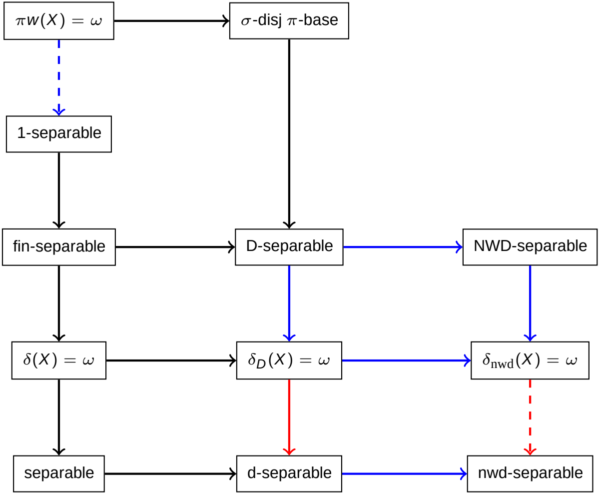
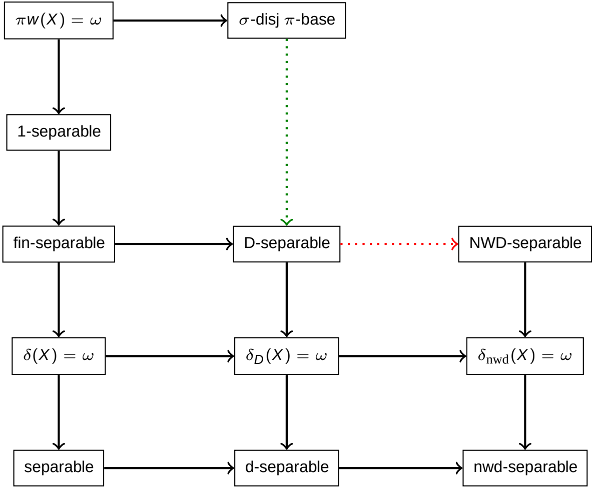


- Moore: there is a dense L-space $X \subset D(2)^{\omega_1}$
- Con(there is uncountable Luzin subspace of $D(2)^{\omega_1}$)

Countable and compact examples



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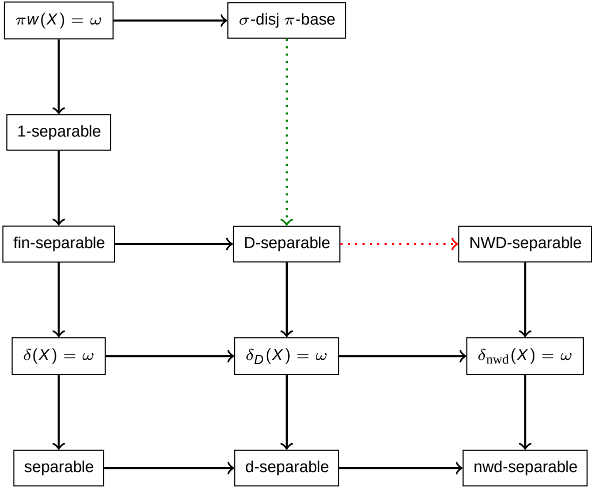

- Con($\exists$ countable, NWD-separable, not D-separable space)?
- Con($\exists$ compact, D-separable, no σ-disjoint π-base)?

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- Thm. If $cof(\mathcal{M}) = \mathfrak{i} = \omega_1$ then there is a countable submaximal space with weight ω_1.

Union of spaces

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Conjecture. The space $X^{d(X)^+}$ is never D-separable.

Thank you!