The joy of elementary submodels

Lajos Soukup

http://www.renyi.hu/∼soukup

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences

Introduction

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- Basic concepts
- Easy applications
- Simplified proofs
- New results and problems

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- $\mathcal{B} \cup \{A\}$ is a larger Δ-system with kernel D than $\mathcal{B}$. **Contradiction.**

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- C is closed $\implies \eta \in C$. **Contradiction** because $\eta \in S$

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- chain of elementary submodels.

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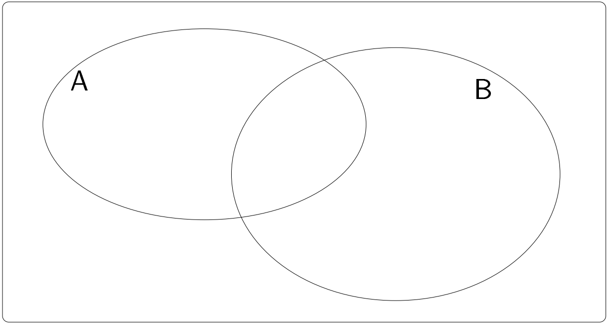

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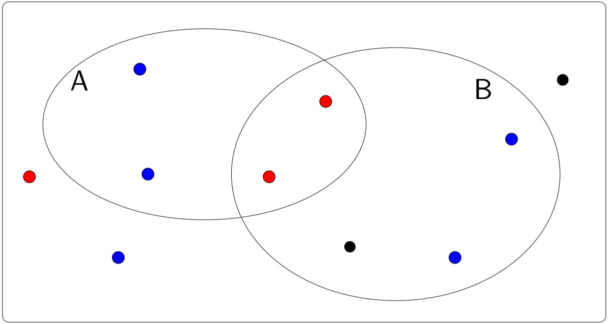

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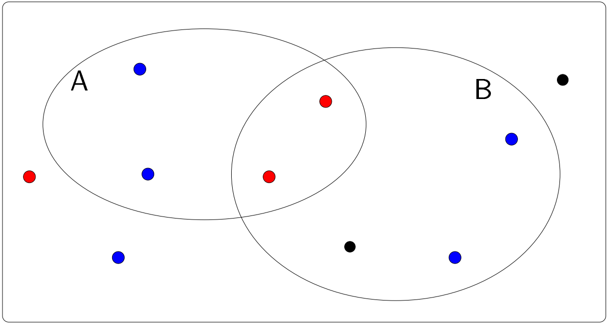

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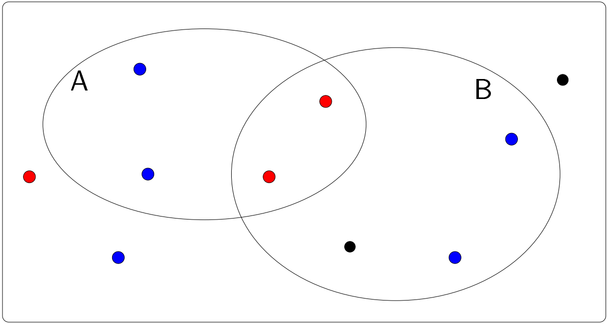

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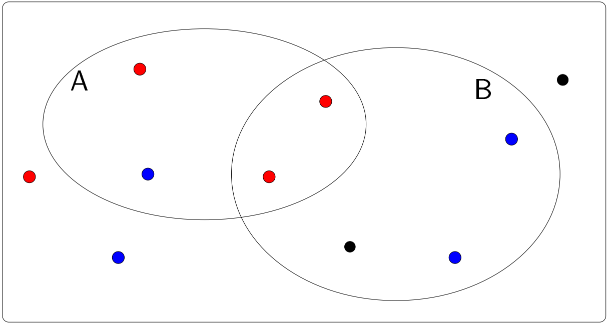

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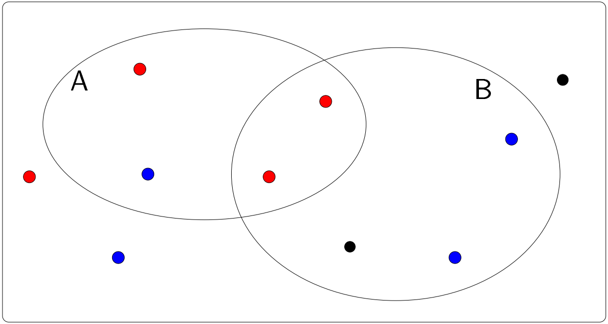

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If $\kappa \not\to [\kappa]^2_\kappa$ *then* $\chi_{CF}[\kappa, \ell, k+1] = \kappa$ *for* $k+1 \leq \ell \leq 2k$.

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Elementary submodels

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Elementary submodels+ Shelah's revised GCH

Shelah's revised GCH

- $\rho^{[\nu]} = \rho$ iff there is a family $\mathcal{B} \subset [\rho]^{\leq\nu}$ of size ρ such that for all $u \in [\rho]^{\nu}$ there is $\mathcal{P} \in [\mathcal{B}]^{<\nu}$ such that $u \subset \cup\mathcal{P}$.

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- **Shelah's Revised GCH theorem:** If $\rho \geq \beth_\omega$, then $\rho^{[\nu]} = \rho$ for each large enough regular $\nu < \beth_\omega$.