Universal locally compact scattered spaces

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Introduction

X is **scattered** iff $I(Y) \neq \emptyset$ for each nonempty $Y \subset X$.

The β^{th} ***Cantor-Bendixson level*** of X is

$$I_\beta(X) = I(X \setminus \cup\{I_\alpha(X) : \alpha < \beta\})$$

The ***reduced height***:

$$\mathrm{ht}^-(X) = \min\{\alpha : I_\alpha(X) \text{ is finite}\}.$$

The ***cardinal sequence*** of X:

$$SEQ(X) = \langle |I_\alpha(X)| : \alpha < \mathrm{ht}^-(X) \rangle.$$

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Regular and 0-dimensional spaces

If X is scattered T_3 then $|X| \le 2^{|I(X)|}$.

Theorem (Juhász-Shelah-S-Szentmiklóssy)

If $s = \langle s(\alpha) : \alpha < \beta \rangle$ is a sequence of infinite cardinals then T. F. A. E.:

(1) $s = SEQ(X)$ for some regular scattered space X,

(2) $|\beta \setminus \alpha| \le 2^{s(\alpha)}$ and $s(\alpha') \le 2^{s(\alpha)}$ for $\alpha < \alpha' < \beta$,

(3) $s = SEQ(X)$ for some 0-dimensional scattered space X.

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Main question

What are the cardinal sequences of (locally) compact scattered spaces (or: superatomic boolean algebras)?

$\mathcal{C}(\alpha) = \{SEQ(X) : X$ compact scattered, $\mathrm{ht}^{-}(X) = \alpha\}$.

Characterize $\mathcal{C}(\alpha)$!

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The bound of ZFC characterizations

Theorem (I. Juhász, B. Weiss, 1996-2005, For countable sequences: R. La Grange, 1977.)

$s \in \mathcal{C}(\omega_1)$ *iff* $s(\alpha) \leq s(\beta)^\omega$ *for each* $\beta < \alpha < \omega_1$.

$\langle\kappa\rangle_\alpha \equiv$ constant κ sequence of length α

GCH $\Rightarrow \langle\omega_1\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \in \mathcal{C}(\omega_1 + 1)$

Theorem (Baumgartner - Shelah, 1987)

$\langle\omega_1\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle \notin \mathcal{C}(\omega_1 + 1)$ *in the Mitchell model*

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Reduction theorem

$$\mathcal{C}_\lambda(\delta) = \{s \in \mathcal{C}(\delta) : s(0) = \lambda = \min[s(\beta) : \beta < \delta)]\}$$

Reduction Theorem (I. Juhász, S, B. Weiss, 2005)

For any δ, for any sequence s of infinite cardinals T.F.A.E

(1) $s \in \mathcal{C}(\delta)$

(2) $s = s_0 {}^\frown s_1 {}^\frown \cdots {}^\frown s_{n-1}$, *where $s_i \in \mathcal{C}_{\lambda_i}(\delta_i)$ such that $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ are infinite cardinals and $\delta = \delta_0 + \cdots + \delta_{n-1}$*

Enough to characterize the classes $\mathcal{C}_\lambda(\delta)$

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Characterization Theorem for $\delta < \omega_2$

Theorem (I. Juhász, S, B. Weiss, 2005)

Under GCH, full characterization of $\mathcal{C}_\lambda(\delta)$ *for* $\delta < \omega_2$.

Some restriction

Assume that GCH holds and $\lambda = cf(\lambda) > \omega$.
Let $s \in \mathcal{C}_\lambda(\delta)$. Clearly $s \in {}^{\delta}\{\lambda, \lambda^+\}$

if $s(\beta) = \lambda$ **then** $s(\beta + 1) = \lambda$

Assume $\kappa < \lambda$ and $\sup \langle \beta_\zeta : \zeta < \kappa \rangle = \beta < \delta$,
if $s(\beta_\zeta) = \lambda$ **for** $\zeta < \kappa$ **then** $s(\beta) = \lambda$.

$$\mathcal{D}_\lambda(\delta) = \{s \in {}^{\delta}\{\lambda, \lambda^+\} : s(0) = \lambda,$$
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$$\mathcal{C}_\lambda(\delta) \subset \mathcal{D}_\lambda(\delta).$$

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Characterization Theorem for $\delta < \omega_2$

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Theorem (Juhász, S, Weiss)

Under GCH, for $\delta < \omega_2$,

(i) ...

(ii) $\mathcal{C}_{\omega_1}(\delta) = \mathcal{D}_{\omega_1}(\delta)$

(ii)' *if* $\lambda = \mathrm{cf}(\lambda) > \omega_1$ *then* $\mathcal{C}_\lambda(\delta) = \{\langle\lambda\rangle_\delta\}$.

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Sequences of length ω_2

Fact (GCH)

$\mathcal{C}_\omega(\omega_2) = \emptyset$ *and* $\mathcal{C}_\lambda(\omega_2) = \{\langle\lambda\rangle_{\omega_2}\}$ *for* $\lambda = \mathrm{cf}(\lambda) > \omega_1$.

$\mathcal{C}_{\omega_1}(\omega_2)$=??

Theorem (Juhász-Shelah-S-Szentmiklóssy)

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Problem

$\langle\omega_1\rangle_{\omega_2} \in \mathcal{C}_{\omega_1}(\omega_2)$ *in ZFC or under GCH*

Sequences of length ω_2

Fact (GCH)

$\mathcal{C}_\omega(\omega_2) = \emptyset$ *and* $\mathcal{C}_\lambda(\omega_2) = \{\langle\lambda\rangle_{\omega_2}\}$ *for* $\lambda = \mathrm{cf}(\lambda) > \omega_1$.

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Theorem (J.C. Martinez, S)

For each $\alpha < \omega_3$ it is consistent with GCH that $\mathcal{C}_{\omega_1}(\alpha) = \mathcal{D}_{\omega_1}(\alpha)$.

Definition

An LCS space X is called ***$\mathcal{C}_\lambda(\alpha)$-universal*** iff $SEQ(X) \in \mathcal{C}_\lambda(\alpha)$ and for each sequence $s \in \mathcal{C}_\lambda(\alpha)$ there is an open subspace Y of X with $SEQ(Y) = s$.

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- For each $s \in \mathcal{D}_{\omega_1}(\delta)$ find a poset P_s such that
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- $|\mathcal{D}_{\omega_1}(\delta)| = \omega_3$***!***
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If $\kappa^{<\kappa} = \kappa$ then $\langle\kappa\rangle_\kappa {}^\frown \langle\kappa^+\rangle_{\kappa^+} \in \mathcal{C}_\kappa(\kappa^+)$.

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Con(GCH + $\langle\aleph_\omega\rangle_{\aleph_\omega} {}^\frown \langle\aleph_{\omega+1}\rangle_{\aleph_{\omega+1}} \in \mathcal{C}_{\aleph_\omega}(\aleph_{\omega+1})$).

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$\neg GCH$

Theorem (Baumgartner, Shelah)

Con(ZFC + $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$*).*

Theorem (Bagaria)

Con(ZFC + ${}^{\omega_2}\{\omega, \omega_1\} \subset \mathcal{C}(\omega_2)$*).*

MA_{ω_2} + there is a Δ-function $2^\omega > \omega_2$.

If $2^\omega = \omega_n$ then $\mathcal{C}_\omega(\omega_2) \subset \{f \in {}^{\omega_2}\{\omega, \ldots, \omega_n\} : f(0) = \omega\}$

Theorem

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Questions

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Is it true that for each cardinal λ and ordinal δ if $\mathcal{C}_\lambda(\delta) \neq \emptyset$ then there is a $\mathcal{C}_\lambda(\delta)$-universal space?

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Is it true under GCH that for each regular cardinal λ and ordinal $\delta < \lambda^{++}$ we have $\mathcal{C}_\lambda(\delta) = \mathcal{D}_\lambda(\delta)$?

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Theorem (Martinez)

Con (ZFC + $\langle\omega\rangle_\alpha \in \mathcal{C}(\alpha)$ for each $\alpha < \omega_3$.)

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If there is a "natural" c.c.c. forcing P such that $\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2)$ in V^P then for each $\alpha < \omega_3$ there is an other "natural" c.c.c. forcing Q such that $\langle\omega\rangle_\alpha \in \mathcal{C}(\alpha)$ in V^Q.

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$\langle\omega\rangle_{\omega_2} \in \mathcal{C}(\omega_2) \overset{?}{\Longrightarrow} \langle\omega\rangle_\alpha \in \mathcal{C}(\alpha)$ *for each* $\alpha < \omega_3$

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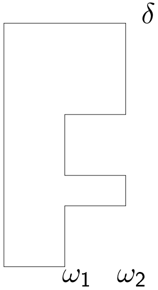

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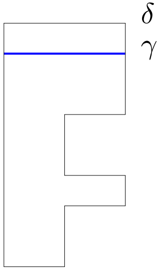

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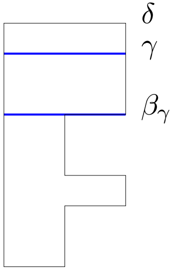

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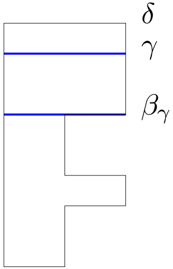

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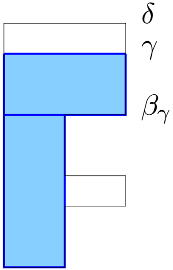

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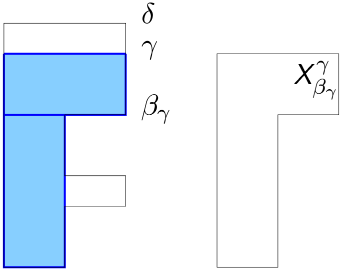

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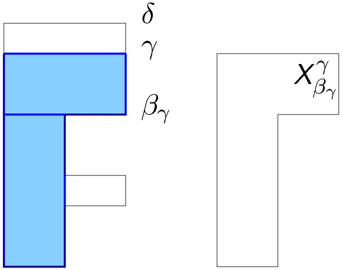

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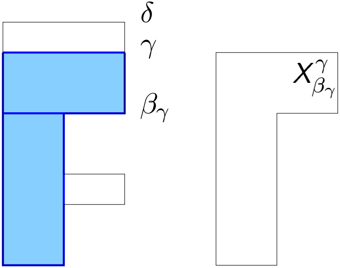

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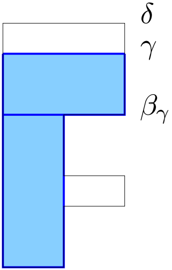

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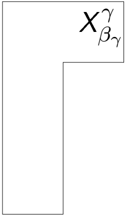

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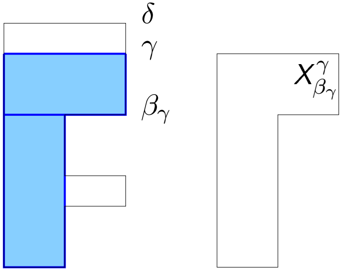

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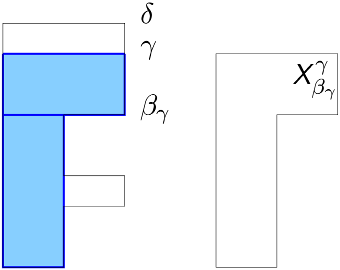

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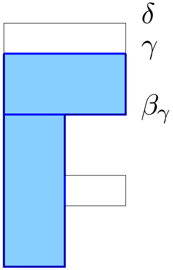

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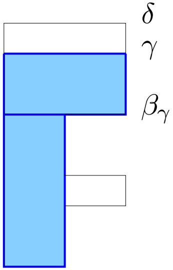

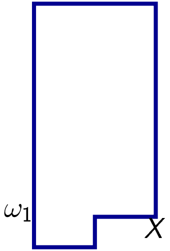

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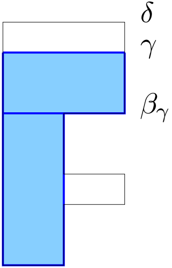

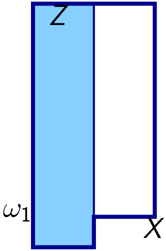

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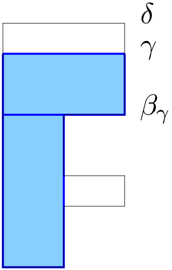

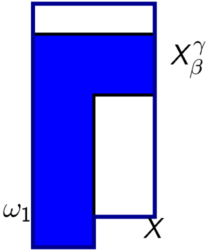

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too many $X^\gamma_{\beta_\gamma}$
amalgamation instead of union

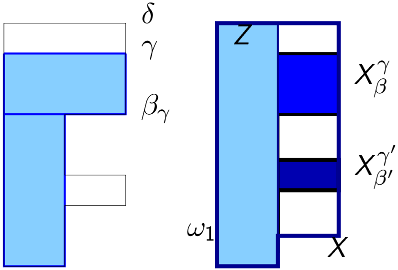
Theorem

$\cup^*\{X^\gamma_\beta : \beta \le \gamma < \delta : cf(\beta) = \omega_1\}$ *is* $\mathcal{C}_{\omega_1}(\delta)$*-universal.*

$\exists X$ LCS, $Z \subset X$ open s.t.
$SEQ(X) = \langle\omega_1\rangle_{\omega_1} {}^\frown \langle\omega_2\rangle_{\delta - \omega_1}$
$SEQ(Z) = \langle\omega_1\rangle_\alpha$
$X^\gamma_\beta \subset X$

$\mathcal{C}_{\omega_1}(\delta) = \mathcal{D}_{\omega_1}(\delta)$

$s \in \mathcal{D}_{\omega_1}(\delta)$
$\gamma < \delta,\ s(\gamma) = \omega_2$
$\beta_\gamma = \min\{\beta \leq \gamma : [\beta, \gamma] \subset s^{-1}\{\omega_2\}\}$
$cf(\beta_\gamma) = \omega_1$
$s^\gamma = \langle\omega_1\rangle_{\beta_\gamma} {}^\frown \langle\omega_2\rangle_{\gamma - \beta_\gamma} \in \mathcal{D}_{\omega_1}(\gamma)$
$\exists\ X^\gamma_{\beta_\gamma}$ s.t. $SEQ(X^\gamma_{\beta_\gamma}) = s^\gamma$?



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$X^\gamma_\beta \subset X \quad X^\gamma_\beta \cap X^{\gamma'}_{\beta'} \subset Z$