Information Theory
Sample Second Midterm

The next two pages contain sketches of solutions so do not turn the page if you want to use these exercises for practicing.

1) Give the definition of a Markov chain.

2) State the converse of the Channel Coding Theorem.

3) Let us have a source with alphabet $X = A, I, K, N, R, T$. Encode the source sequence

$$TANARIKARIKARIKA$$

with the Lempel-Ziv-Welch algorithm. (The dictionary originally contains the codewords (i.e., the indices) for the one character sequences: 1 for A, 2 for I, 3 for K, 4 for N, 5 for R, 6 for T.) Give both the code and the dictionary we have after the whole string above is encoded. (When two-digit numbers appear in the encoding as the index of some subsequence in the dictionary, then put those two digits into brackets to indicate that they mean one index.) .

4) Let the density function of the random variable X be $\frac{3}{8}x^2$ for $x \in [0,2]$ and 0 outside this interval. We quantize this source variable with a 2-level quantizer. Starting with initial quantization levels $x_1 = \frac{1}{2}, x_2 = \frac{3}{2}$ perform one iteration of the Lloyd-Max algorithm and give the new quantization levels and the quantization intervals belonging to them after this iteration.

5)Let X be a source whose output is the state of a stationary Markov chain that has three possible states A, B, C and the following transition probabilities.

$$P(A|A) = \frac{1}{2}, \quad P(B|A) = \frac{1}{4}, \quad P(C|A) = \frac{1}{4},$$
$$P(A|B) = \frac{1}{3}, \quad P(B|B) = \frac{1}{3}, \quad P(C|B) = \frac{1}{3},$$
$$P(A|C) = \frac{1}{4}, \quad P(B|C) = \frac{1}{4}, \quad P(C|C) = \frac{1}{2}.$$

Determine the entropy of this source (if it exists).

6) We have a channel with identical input and output alphabet of three letters that we denote by v, w, z. When v is sent the received letter is v with probability $\frac{2}{3}$ and it is w with probability $\frac{1}{3}$. When w is sent then the output can be either of v, w, z each having conditional probability $\frac{1}{3}$. When z is sent it becomes w at the output with probability $\frac{1}{3}$ and it will be z with probability $\frac{2}{3}$. Determine the capacity of this channel.

Sketches of solutions

1. A stochastic process $Z = Z_1, Z_2, \dots$ is Markov, if for every k we have $P(Z_k|Z_1, \dots, Z_{k-1}) = P(Z_k|Z_{k-1})$. The variables $Z_1, Z_2, \dots$ of such a stochastic process form a Markov chain.

2. For any sequence of codes with length n and number of codewords at least 2^{nR} it is true that if using these codes for communication over a channel with capacity C we have average error probability tending to zero as n goes to infinity, then we must have $R \leq C$.

3. Following the Lempel-Ziv-Welch algorithm we obtain the code

$$6, 1, 4, 1, 5, 2, 3, (10), (12), (14), (13)$$

and the dictionary

$$1: A;\;\; 2: I;\;\; 3: K;\;\; 4: N;\;\; 5: R;\;\; 6: T;\;\; 7: TA;\;\; 8: AN; 9:\;\; NA;\;\; 10: AR;$$

$$11: RI;\;\; 12: IK;\;\; 13: KA;\;\; 14: ARI;\;\; 15: IKA;\;\; 16: ARIK.$$

4. The two original quantization intervals we get are $(-\infty, 1)$ and $[1, \infty)$, but since $f(X)$ is 0 outside $[0, 2]$, it is enough to consider $[0, 1)$ and $[1, 2]$. So we have to calculate

$$\frac{\int_0^1 x \frac{3}{8} x^2 dx}{\int_0^1 \frac{3}{8} x^2 dx}$$

and

$$\frac{\int_1^2 x \frac{3}{8} x^2 dx}{\int_1^2 \frac{3}{8} x^2 dx}.$$

The first one of these gives

$$\frac{\left[\frac{3}{32} x^4\right]_0^1}{\left[\frac{3}{24} x^3\right]_0^1} = \frac{\frac{3}{32}}{\frac{3}{24}} = \frac{3}{4}.$$

The second one gives

$$\frac{\left[\frac{3}{32} x^4\right]_1^2}{\left[\frac{3}{24} x^3\right]_1^2} = \frac{\frac{3}{2} - \frac{3}{32}}{1 - \frac{3}{24}} = \frac{45}{28}.$$

Thus the new quantization levels are $\frac{3}{4}$ and $\frac{45}{28}$, while the new quantization intervals are $(-\infty, a], (a, \infty)$ with $a = \frac{\frac{3}{4} + \frac{45}{28}}{2} = \frac{33}{28}$.

5. First we need to calculate the stationary distribution. Denoting the probabilities of state A, B, and C with a, b, and c, respectively, we have

$$a = \frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c,$$

$$b = \frac{1}{4}a + \frac{1}{3}b + \frac{1}{4}c,$$

$$c = \frac{1}{4}a + \frac{1}{3}b + \frac{1}{2}c.$$

Using the first and third (could be other two but that is the easiest) equations and

$$a + b + c = 1$$

we obtain $a = c = \frac{4}{11}$ and $b = \frac{3}{11}$. So the requested entropy is (using that the Markov chain is stationary)

$$H(X) = H(X_2|X_1) = \frac{4}{11}H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) + \frac{3}{11}H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \frac{4}{11}H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) =$$

$$\frac{3}{11}\log 3 + \frac{12}{11}.$$

6. We need to calculate $C = \max I(X,Y) = \max\{H(Y) - H(Y|X)\}$. Let the input distribution (which we should choose optimally) be given by probability values p, q, r for the input letters v, w, z, respectively. Then $H(Y|X) = (p + r)h(1/3) + q\log 3 = \log 3 - (p+r)\frac{2}{3} = \log 3 - (1-q)\frac{2}{3}$. This is smallest when $q = 0$. At the same time $H(Y) \leq \log 3$ and this can be attained with $p = r = \frac{1}{2}$ and $q = 0$, because $Prob(v) = Prob(z) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ and $Prob(w) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$, as well. So $I(X,Y) = H(Y) - H(Y|X)$ is maximized at $q = 0, p = r = \frac{1}{2}$ and then its value is equal to $\log 3 - (\log 3 - \frac{2}{3}) = \frac{2}{3}$. So the channel capacity is this value:

$$C = \frac{2}{3}.$$

Note, that this means that we are best off if we do not use the input letter w and then we essentially have a binary erasure channel with capacity $\frac{2}{3}$.