Homework 3

due February 26, 2025.

1. Prove that every finite simple graph G contains a (not necessarily induced) subgraph which is bipartite and contains at least half of the edges of G.

2. Prove that it is true for every positive integer k that a k-regular bipartite graph always contains a perfect matching.

   (Reminder: A graph is called k-regular if all of its vertices have degree k.)

3. What is the maximum possible value of the ratio

   $$\frac{\tau(G)}{\nu(G)},$$

   where the maximization is over all finite simple graphs G?

   (As usually, $\nu(G)$ denotes the number of edges in a largest matching in G, while $\tau(G)$ denotes the size of a smallest set of vertices U satisfying that every edge of G has at least one of its endvertices in U.)

4. Prove that if A is a doubly stochastic matrix, then its permanent $\mathrm{Per}(A)$ is strictly positive.

   Definitions needed to understand this problem:

   An $n \times n$ matrix $M = (m_{i,j})_{i,j=1}^{n}$ is doubly stochastic if all its entries are non-negative and all row sums and all column sums are 1. (That is $\sum_{i=1}^{n} m_{i,j} = 1$ for all j and $\sum_{j=1}^{n} m_{i,j} = 1$ for all i. Note that the latter conditions already imply that the number of rows and columns must be the same.)

   The permanent $\mathrm{Per}(M)$ of a square matrix $M = (m_{i,j})_{i,j=1}^{n}$ is defined as

   $$\mathrm{Per}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)},$$

   where S_n denotes the set of all permutations of the numbers in $\{1, 2, \ldots, n\}$.

5. Let k, n be positive integers satisfying $k < n/2$. Prove that for every k-element subset A of $[n] = \{1, \ldots, n\}$ one can add an element from $[n] \setminus A$ to A in such a way that the so obtained $\binom{n}{k}$ $(k+1)$-element subsets of $[n]$ are all different.