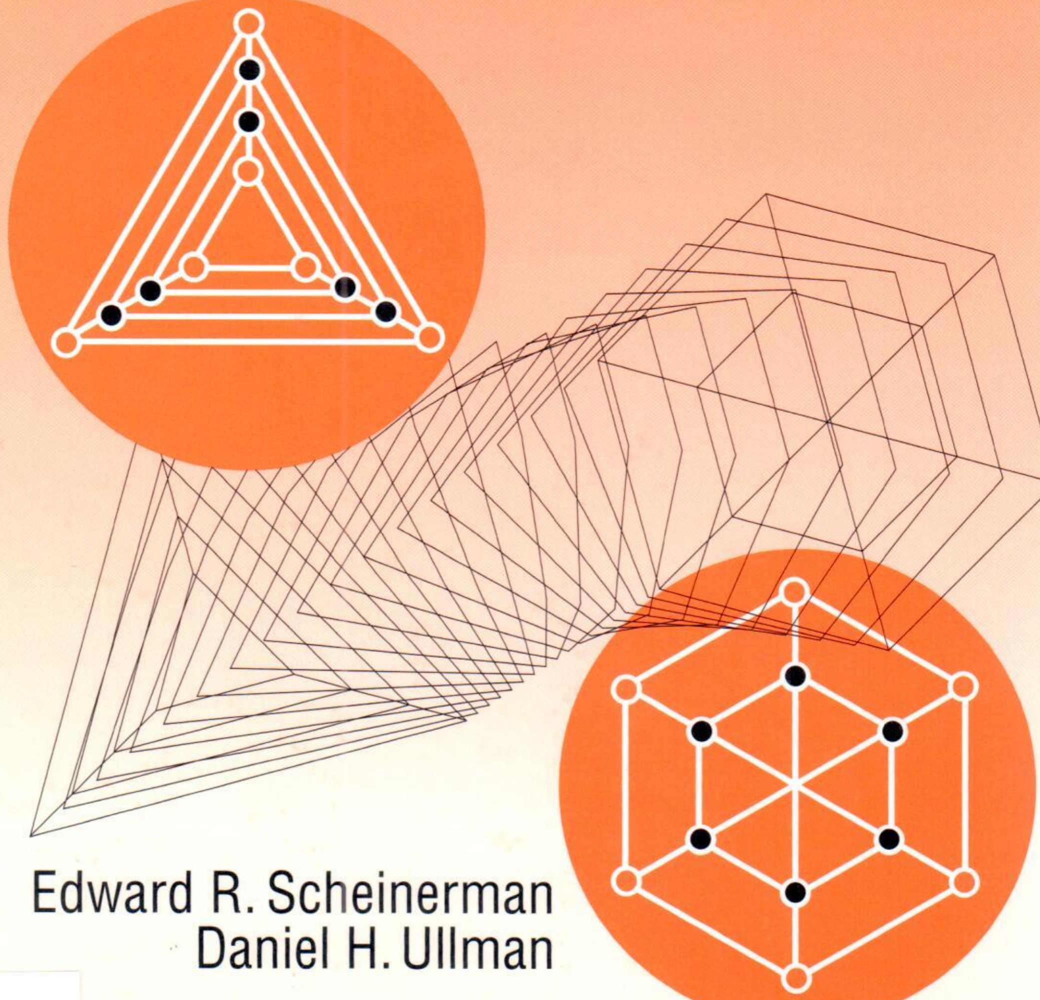
Fractional Graph Theory

A Rational Approach to the Theory of Graphs



Edward R. Scheinerman
Daniel H. Ullman

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Preface

Graphs model a wide variety of natural phenomena, and it is the study of these phenomena that gives rise to many of the questions asked by pure graph theorists. For example, one motivation for the study of the chromatic number in graph theory is the well-known connection to scheduling problems. Suppose that an assortment of committees needs to be scheduled, each for a total of one hour. Certain pairs of committees, because they have a member in common, cannot meet at the same time. What is the length of the shortest time interval in which all the committees can be scheduled?

Let G be the graph whose vertices are these committees, with an edge between two committees if they cannot meet at the same time. The standard answer to the scheduling question is that the length of the shortest time interval is the chromatic number of G. As an illustration, suppose that there are 5 committees, with scheduling conflicts given by the graph in Figure A. Since

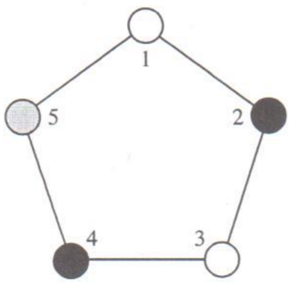

Figure A: The graph C_5, colored with three colors.

G can be colored with 3 colors, the scheduling can be done in 3 hours, as is illustrated in Figure B.

It is a widely held misconception that, since the chromatic number of G is 3, the schedule in Figure B cannot be improved. In fact, the 5 committees can be scheduled in two-and-a-half hours, as is illustrated in Figure C.

All that is required is a willingness to allow one committee to meet for half an hour, to interrupt their meeting for a time, and later to reconvene for the remaining half hour. All that is required is a willingness to break one whole hour into *fractions* of an hour—to break a discrete unit into *fractional*

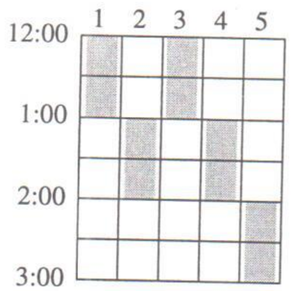

Figure B: A schedule for the five committees.

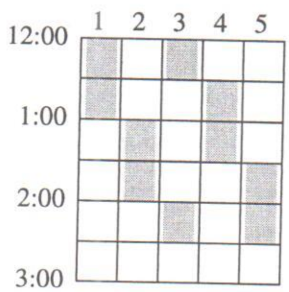

Figure C: An improved schedule for the five committees.

parts. The minimum length of time needed to schedule committees when interruptions are permitted is not the chromatic number of G but the less well-known *fractional chromatic number* of G.

This example illustrates the theme of this book, which is to uncover the rational side of graph theory: How can integer-valued graph theory concepts be modified so they take on nonintegral values? This "fractionalization" bug has infected other parts of mathematics. Perhaps the best-known example is the fractionalization of the factorial function to give the gamma function. Fractal geometry recognizes objects whose dimension is not a whole number [126]. And analysts consider fractional derivatives [132]. Some even think about fractional partial derivatives!

We are not the first to coin the term *fractional graph theory*; indeed, this is not the first book on this subject. In the course of writing this book we

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Rationalization

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[1]More recently, Berge devotes
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3

Fractional Coloring

3.1 Definitions

The most celebrated invariant in all of graph theory is the *chromatic number*. Recall that an *n-coloring* of a graph G is an assignment of one of n colors to each vertex so that adjacent vertices receive different colors. The *chromatic number* of G, denoted $\chi(G)$, is the least n for which G has an n-coloring.

The *fractional chromatic number* is defined as follows. A *b-fold coloring* of a graph G assigns to each vertex of G a set of b colors so that adjacent vertices receive disjoint sets of colors. We say that G is *a: b-colorable* if it has a b-fold coloring in which the colors are drawn from a palette of a colors. We sometimes refer to such a coloring as an a: b-coloring. The least a for which

G has a b-fold coloring is the *b-fold chromatic number* of G, denoted $\chi_b(G)$. Note that $\chi_1(G) = \chi(G)$.

Since $\chi_{a+b}(G) \le \chi_a(G) + \chi_b(G)$, we define the *fractional chromatic number* to be

$$\chi_f(G) = \lim_{b\to\infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}.$$

(See §A.4 on page 188.)

We can also use the methods of Chapter 1 to describe the fractional chromatic number. Associate with a graph G its *vertex-independent set hypergraph* $\mathcal{H}$ defined as follows. The vertices of $\mathcal{H}$ are just the vertices of G, while the hyperedges of $\mathcal{H}$ are the independent sets of G. Then $k(\mathcal{H})$ is the minimum number of independent sets of vertices needed to cover $V(G)$. Since a subset of an independent set is again independent, this is equivalent to the minimum number of independent sets needed to *partition* $V(G)$, i.e., the chromatic number. (See also exercise 7 on page 17.) Thus $k(\mathcal{H}) = \chi(G)$. Furthermore, the b-fold chromatic number of G is just the b-fold covering number of $\mathcal{H}$ and so $\chi_f(G) = k_f(\mathcal{H})$.

We know from Corollary 1.3.1 on page 5 that $\chi_f(G)$ is a rational number and from Corollary 1.3.2 that there is a b so that $\chi_f(G) = \chi_b(G)/b$. If G has no edges, then $\chi_f(G) = 1$. Otherwise, $\chi_f(G) \ge 2$.

Proposition 3.1.1 *For any graph G, $\chi_f(G) \ge \nu(G)/\alpha(G)$. Furthermore, if G is vertex-transitive, then equality holds.*

Proof. Immediate from Proposition 1.3.4 on page 7. □

What is the dual notion? There is a natural interpretation of $p_f(\mathcal{H})$ (where $\mathcal{H}$ is the vertex-independent set incidence hypergraph of G). Note that $p(\mathcal{H})$ is the maximum number of vertices no two of which are together in an independent set. Stated another way, $p(\mathcal{H})$ is the maximum size of a clique, so $p(\mathcal{H}) = \omega(G)$. Thus $p_f(\mathcal{H})$ is the *fractional clique number* of G, denoted $\omega_f(G)$. By duality, $\chi_f(G) = \omega_f(G)$.

The fractional clique number can also be defined as follows. For a graph G and positive integers a, b, we call a multiset of vertices K an *a:b-clique* if $|K| = a$ and if for every independent set of vertices S the number of vertices of $K \cap (aS)$ (counting multiplicities) is at most b. The b-fold clique number of G, denoted $\omega_b(G)$, is the largest a such that G has an a:b-clique, and then

$$\omega_f(G) = \lim_{b\to\infty} \frac{\omega_b(G)}{b} = \sup_b \frac{\omega_b(G)}{b}.$$

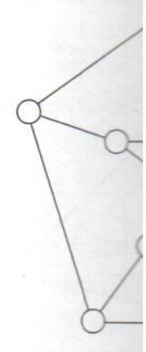

Figure 3.1. T

A graph G is called *perfect* i
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page 72.

Proposition 3.1.2 $\chi_f(C_{2m+1}) =$

Proof. Since cycles are vertex-tran
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3.2 Homomorphisms and

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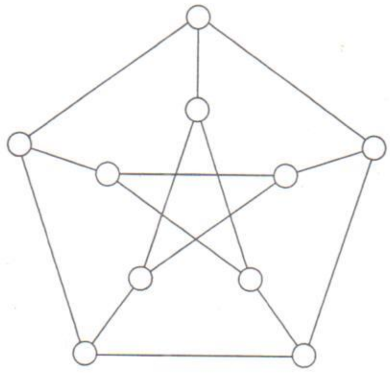

Figure 3.1. The Petersen graph $K_{5:2}$.

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs of H. Perfection becomes trivial in fractional graph theory: for all graphs G, $\chi_f(H) = \omega_f(H)$ for all induced subgraphs H of G! See also exercise 9 on page 72.

Proposition 3.1.2 $\chi_f(C_{2m+1}) = 2 + (1/m)$

Proof. Since cycles are vertex-transitive and $\alpha(C_{2m+1}) = m$, the result follows from Proposition 3.1.1 on the facing page. □

3.2 Homomorphisms and the Kneser graphs

Given positive integers a and b, define a graph $K_{a:b}$ as follows: the vertices are the b-element subsets of a fixed a-element set. There is an edge between two of these vertices if they are disjoint sets. The graphs $K_{a:b}$ are known as the *Kneser graphs* and they play a central role in the theory of the fractional chromatic number. As an illustration, $K_{5:2}$ is pictured in Figure 3.1.

We restrict attention to the case where $a \geq 2b$, since otherwise $K_{a:b}$ has no edges. Note that $K_{a:1} = K_a$, the complete graph on a vertices. The graph $K_{a:2}$ is just $\overline{L(K_a)}$, the complement of the line graph of the complete graph on a vertices.

Suppose that G and H are graphs. A *homomorphism* $\phi : G \to H$ is a mapping from $V(G)$ to $V(H)$ such that $\phi(v)\phi(w) \in E(H)$ whenever $vw \in E(G)$.

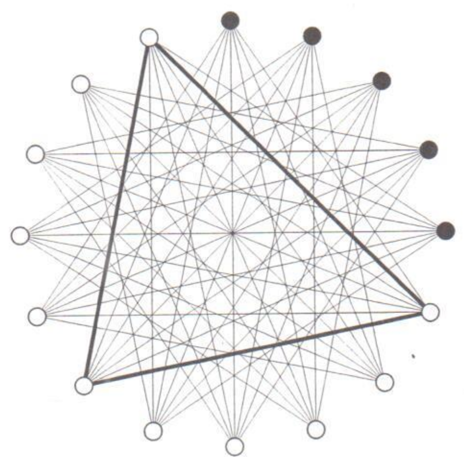

Figure 3.2. The graph $G_{16,5}$ from Proposition 3.2.2. Note that $\alpha(G_{16,5}) = 5$ (black vertices) and $\omega(G_{16,5}) = \lfloor 16/5 \rfloor = 3$ (thick edges). See also exercise 1 on page 72.

There is a simple connection between graph coloring and graph homomorphisms, namely, a graph G is n-colorable if and only if there is a homomorphism $\phi: G \to K_n$.

This easily generalizes to $a{:}\,b$-colorings and motivates our study of Kneser graphs.

Proposition 3.2.1 *A graph G has an $a{:}\,b$-coloring if and only if there is a graph homomorphism $\phi : G \to K_{a:b}$.* □

We know that χ_f can take only rational values and that $\chi_f(G) = 0$, $\chi_f(G) = 1$, or $\chi_f(G) \geq 2$ (see exercises 2 and 3 on page 72). In fact, all such values are actually achieved.

Proposition 3.2.2 *Let a and b be positive integers with $a \geq 2b$. Let $G_{a,b}$ be the graph with vertex set $V(G) = \{0, 1, \ldots, a-1\}$. The neighbors of vertex v are $\{v+b, v+b+1, \ldots, v+a-b\}$ with addition modulo a.*

Then $\chi_f(G_{a,b}) = a/b$ and $\chi_b(G_{a,b}) = a$.

Think of the vertices of $G_{a,b}$ as equally spaced points around a circle, with an edge between two vertices if they are not too near each other; see Figure 3.2.

Proof. Note that $G_{a,b}$ has
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Proof. Note that $G_{a,b}$ has a vertices and is vertex-transitive. The maximum independent set of $G_{a,b}$ are all those of the form $\{i+1, i+2, \ldots, i+b\}$, so $\alpha(G_{a,b}) = b$ (exercise 1 on page 72). By Proposition 3.1.1 on page 42, $\chi_f(G_{a,b}) = a/b$. This implies $\chi_b(G) \geq a$ so it remains to exhibit an $a{:}b$-coloring of $G_{a,b}$. Let the colors be $\{0, 1, \ldots, a-1\}$ and assign to vertex v the color set $\{v, v+1, \ldots, v+b-1\}$ with addition modulo a. It is easy to check that if $vw \in E(G_{a,b})$ then the color sets assigned to v and w are disjoint. □

Another graph whose fractional chromatic number is a/b is the Kneser graph $K_{a:b}$. There is a close connection between this fact and the following well-known theorem [55] of extremal set theory, which has a simple phrasing in terms of the independence number of the Kneser graphs.

Theorem 3.2.3 (Erdős-Ko-Rado) *If a and b are positive integers with $a > 2b$, then*

$$\alpha(K_{a:b}) = \binom{a-1}{b-1}.$$

We prove this theorem below and, in fact, show that it is equivalent to the following proposition.

Proposition 3.2.4 $\chi_f(K_{a:b}) = a/b$.

We provide two proofs, both very short. The first proof uses the Erdős-Ko-Rado theorem 3.2.3. The second uses composition of graph homomorphisms and is independent of Theorem 3.2.3. We may thus use Proposition 3.2.4 to prove the Erdős-Ko-Rado theorem without circularity.

Proof 1. The Kneser graph $K_{a:b}$ has $\binom{a}{b}$ vertices and is vertex-transitive. By Theorem 3.2.3, $\alpha(K_{a:b}) = \binom{a-1}{b-1}$. Hence, by Proposition 3.1.1

$$\chi_f(K_{a:b}) = \binom{a}{b} \Big/ \binom{a-1}{b-1} = a/b. \qquad \square$$

Proof 2. Let $G_{a,b}$ be the graph of Proposition 3.2.2 on the facing page. That result tells us that there is a homomorphism $\phi : G_{a,b} \to K_{a:b}$. Suppose that $K_{a:b}$ has a $c{:}d$-coloring; in other words, suppose that there is a homomorphism $\psi : K_{a:b} \to K_{c:d}$. Then $\psi \circ \phi$ would be a homomorphism from $G_{a,b}$ to $K_{c:d}$, so $G_{a,b}$ would have a $c{:}d$-coloring. Hence $\frac{c}{d} \geq \chi_f(G_{a,b}) = \frac{a}{b}$. We conclude that the natural $a{:}b$-coloring of $K_{a:b}$ (i.e., the identity coloring) is optimal. □