Around the multivariate Bernstein inequality on convex bodies

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A geometric approach to prove a Bernstein-type inequality for a compact, convex body $K \subset \mathbb{R}^d$ is the inscribed ellipse method, first used in this context by Y. Sarantopoulos in 1991. For convex *centrally symmetric* sets the inscribed ellipse method gives a sharp Bernstein type inequality. For not necessarily symmetric convex bodies, our best estimates use the so-called generalized Minkowski functional, and are within a $\sqrt{2}$ constant factor of the truth, but a seemingly natural conjecture, formulated jointly with Y. Sarantopoulos in 2002, fails, as was shown by an insightful construction of N. Naidenov in 2006.

Pluripotential theory is a non-linear generalization to $\mathbb{C}^d$ of potential theory in $\mathbb{C}$. The basic objects of study are plurisubharmonic (psh) functions and closed positive currents, and the complex Monge-Ampére operator is a nonlinear generalization of the Laplacian. One can define the analogue of the classical Green function, called the Siciak-Zaharjuta extremal function, V_K, associated to a compact set $K \subset \mathbb{C}^N$. One can use V_K to prove a Bernstein-type inequality for multivariate polynomials. This was done by M. Baran in 1994-1995.

V_K is an example of a maximal function in $\mathbb{C}^d \setminus K$; maximal functions are multivariate generalizations of harmonic functions, but unlike the latter they are not necessarily smooth. One reason for a psh function u to be maximal on a domain is the local existence, through each point, of a one-dimensional complex submanifold on which u is harmonic. For the function V_K, this property is known to hold if K is the closure of a smooth, strictly linearly convex domain in $\mathbb{C}^d$ (Lempert). We prove (joint work with D. Burns, N. Levenberg and S. Ma'u) that it also holds if K is a convex body in $\mathbb{R}^d$. It turns out that the manifolds on which V_K is harmonic are complex ellipses that intersect K in a real ellipse. These ellipses have certain extremal properties; in many cases they give a continuous foliation of $\mathbb{C}^d \setminus K$.

Both the pluripotential theory and the inscribed ellipse methods give a sharp estimate for symmetric convex bodies. In the case of the standard simplex in $\mathbb{R}^d$, the exact yields of both methods have been computed (jointly with L. Milev, in 2005), and were found to be equivalent.

The equivalence of the Bernstein factors obtained by the two methods is no coincidence. We show (joint work with D. Burns, N. Levenberg and S. Ma'u) that the ellipses that give the best bound for the inscribed ellipse method are identical to the ellipses whose complexifications give the one-dimensional manifolds on which V_K is harmonic. This is used to show that indeed, the two methods are equivalent for all convex bodies $K \subset \mathbb{R}^d$.