Independence of Solution Sets in Additive Number Theory

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1. INTRODUCTION

Let A be a strictly increasing sequence of positive integers. Let $2A$ denote the set of all integers of the form $n = a + a'$, where $a, a' \in A$. If $n \in 2A$ for all sufficiently large n, then A is an *asymptotic basis of order 2*, or, simply, a *basis*. Let $r_A(n)$ denote the number of representations of n in the form $n = a + a'$, where $a, a' \in A$ and $a \leq a'$. An old conjecture of Erdös and Turán [2] states that if A is a basis, then $r_A(n)$ is unbounded. Let

$$S_A(n) = \{a \in A \mid n - a \in A, n \neq 2a\}$$

denote the *solution set of* n. Clearly, $S_A(1) = S_A(2) = \varnothing$ and $S_A(n) \subseteq [1, n-1]$. Let $|S|$ denote the cardinality of the set S. Then

$$|S_A(n)| = \begin{cases} 2r_A(n) & \text{if } n/2 \notin A \\ 2r_A(n) - 2 & \text{if } n/2 \in A. \end{cases}$$

Let Ω denote the space of all strictly increasing sequences of positive integers. Let $p(1), p(2), \ldots, p(n), \ldots$ be any sequence of real numbers in the unit interval $[0, 1]$. Let

$$E_n = \{A \in \Omega \mid n \in A\}$$

denote the set of all sequences $A \in \Omega$ that contain n. Erdös and Rényi [1] constructed a probability measure μ on Ω such that

(i) $\mu(E_n) = p(n)$, and
(ii) the events $E_1, E_2, \ldots$ are independent.

Choosing $p(1) = \frac{1}{2}$ and $p(n) = \alpha((\log n)/n)^{1/2}$ for $n \geq 2$, they proved that for almost all $A \in \Omega$ there exist constants $0 < c < c'$ such that

$$c \log n < r_A(n) < c' \log n$$

for all sufficiently large n. This result solved a problem of Sidon [5], who asked if there existed a basis A such that

$$\lim_{n \to \infty} r_A(n)/n^{\varepsilon} = 0$$

for every $\varepsilon > 0$. Halberstam and Roth [3] contains a careful exposition of the Erdös–Rényi method.

In this paper we consider probability spaces Ω defined by a sequence of real numbers $p(n) \in [0, 1]$ satisfying the following condition: There exist real numbers α, β, γ with $\alpha > 0$ and

$$\tfrac{1}{3} < \gamma \leq \tfrac{1}{2} \tag{1}$$

such that

$$p(n) \leq \frac{\alpha \log^{\beta}(n+1)}{n^{\gamma}} \tag{2}$$

for all $n \geq 1$. We shall prove that for almost all sequences $A \in \Omega$ the solution sets $S_A(n)$ are "independent" in the sense that $|S_A(m) \cap S_A(n)|$ is bounded for all $n > m$. If $p(n)$ satisfies (1) and (2), then for almost all $A \in \Omega$ and for all but finitely many pairs (m, n) with $n > m$,

$$|S_A(m) \cap S_A(n)| \leq 2/(3\gamma - 1).$$

In particular, if $p(1) = \frac{1}{2}$ and $p(n) = \alpha((\log n)/n)^{1/2}$ for $n \geq 2$, then $\gamma = \frac{1}{2}$ and $|S_A(m)| > c \log m$, but

$$|S_A(m) \cap S_A(n)| \leq 4$$

for almost all $A \in \Omega$ and for all but finitely many pairs (m, n) with $n > m$.

2. Notation

We use the following notation. Let m and n be positive integers with $m < n$. Suppose

$$T \subseteq S_A(m) \cap S_A(n) \tag{3}$$

and $|T| = t$. Then $T \subseteq [1, m-1]$. If $b \in T$, then $m - b \in A$ and $n - b \in A$. Also, $b \neq m/2$ and $b \neq n/2$. The set T determines three subsets U, V, W of $[1, (m-1)/2]$ in the following way:

$$U = \{a \in [1,(m-1)/2] \mid a \in T, m - a \notin T\} = \{a_1, a_2, \ldots, a_u\}, \tag{4}$$

$$V = \{a \in [1,(m-1)/2] \mid a \in T, m - a \in T\} = \{a_{u+1}, \ldots, a_{u+v}\}, \tag{5}$$

$$W = \{a \in [1,(m-1)/2] \mid a \notin T, m - a \in T\} = \{a_{u+v+1}, \ldots, a_{u+v+w}\}, \tag{6}$$

where $|U| = u$, $|V| = v$, and $|W| = w$. The sets U, V, W are pairwise disjoint and determine T, since

$$T = U \cup V \cup \{m - a \mid a \in V \cup W\}. \tag{7}$$

Clearly, $|T| = u + 2v + w$.

The sets U, V, W determine three new sets X, Y, Z. Define

$$X = U \cup V \cup W = \{a_1, a_2, \ldots, a_{u+v+w}\}. \tag{8}$$

Then $|X| = x = u + v + w$, and $X \subseteq [1, (m-1)/2]$. Define

$$Y = \{m - a \mid a \in X\}. \tag{9}$$

Then $|Y| = x$ and $Y \subseteq [(m+1)/2, m-1]$. Define

$$Z = \{n - b \mid b \in T\}. \tag{10}$$

Then $|Z| = t$ and $Z \subseteq [n - m + 1, n - 1]$. Clearly, $X \cap Y \neq \emptyset$ and

$$X \cup Y \cup Z \subseteq A. \tag{11}$$

Conversely, let $X \subseteq [1, (m-1)/2]$ and let $X = U \cup V \cup W$ be a partition of X into three pairwise disjoint sets. Define T, Y, Z by (7), (9), (10). Then $T \subseteq S_A(m) \cap S_A(n)$ if and only if $X \cup Y \cup Z \subseteq A$.

3. Results

THEOREM 1. *Let Ω be the space of all strictly increasing sequences of positive integers with the probability measure μ defined by a sequence $p(n)$ satisfying (1) and (2). For almost all $A \in \Omega$ and for all but finitely many pairs (m, n) of positive integers with $n \geq 2m$,*

$$|S_A(m) \cap S_A(n)| \leq 2/(3\gamma - 1).$$

Proof. Let $t > 2/(3\gamma - 1)$ and $n \geq 2m$. Define

$$\mu_t(m,n) = \mu(\{A \in \Omega \,\big|\, |S_A(m) \cap S_A(n)| \geq t\}). \tag{12}$$

We shall prove that

$$\sum_{m=1}^{\infty} \sum_{n=2m}^{\infty} \mu_t(m,n) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{n=2^k m}^{2^{k+1}m-1} \mu_t(m,n) < \infty.$$

Then it follows from the Borel–Cantelli lemma that

$\mu(\{A \in \Omega \,\|\, S_A(m) \cap S_A(n)| \geq t$ for infinitely many pairs (m,n) with $n \geq 2m\}) = 0$.

This is precisely Theorem 1.

First we estimate $\mu_t(m,n)$. Fix a partition of the integer t of the form $t = u + 2v + w$. Let $x = u + v + w$. Let $X \subseteq [1,(m-1)/2]$ satisfy $|X| = x$. There are $x!/u!v!w!$ partitions of X into three pairwise disjoint sets U, V, W such that $|U| = u$, $|V| = v$, $|W| = w$. Fix a partition of X in the form $X = U \cup V \cup W$, and define T, Y, Z by (7), (9), (10). Then (3) holds if and only if (11) holds. Moreover, every set T satisfying (3) is of the form (7) for some partition of t in the form $t = u + 2v + w$ and some partition of X in the form $X = U \cup V \cup W$, where $X \subseteq [1,(m-1)/2]$ and $|X| = x$. Therefore,

$$\mu_t(m,n) = \sum_t{}^{(3)} \sum_X{}^{(2)} \sum_{U,V,W}{}^{(1)} \mu(\{A \in \Omega \,|\, X \cup Y \cup Z \subseteq A\}), \tag{13}$$

where $\sum_t^{(3)}$ denotes the sum over all partitions of t in the form $t = u + 2v + w$, $\sum_X^{(2)}$ denotes the sum over all subsets $X \subseteq [1,(m-1)/2]$ satisfying $|X| = x = u + v + w$, and $\sum_{U,V,W}^{(1)}$ denotes the sum over all partitions of X in the form $X = U \cup V \cup W$, where $|U| = u$, $|V| = v$, $|W| = w$.

Define T by (7). Then the sets U, V, W satisfy (4), (5), (6). Define the sets Y and Z by (9) and (10). Since $n \geq 2m$, it follows that $n - m + 1 > m$, hence $(X \cup Y) \cap Z = \varnothing$, and so the sets X, Y, Z are pairwise disjoint. Therefore, using (2), we obtain

$$\begin{aligned}
&\mu(\{A \in \Omega \,|\, X \cup Y \cup Z \subseteq A\}) \\
&\quad = \prod_{i=1}^{x} p(a_i) \prod_{i=1}^{x} p(m - a_i) \prod_{i=1}^{u+v} p(n - a_i) \prod_{i=u+1}^{x} p(n - m + a_i) \\
&\quad \leq (\alpha \log^{\beta} n)^{2x+t} \prod_{i=1}^{x} \frac{1}{a_i^{\gamma}} \prod_{i=1}^{x} \frac{1}{(m - a_i)^{\gamma}} \prod_{i=1}^{u+v} \frac{1}{(n - a_i)^{\gamma}} \prod_{i=u+1}^{x} \frac{1}{(n - m + a_i)^{\gamma}}.
\end{aligned}$$

Since $m - a_i > m/2$, $n - a_i > n - m$, and $n - m + a_i > n - m$, we obtain

$$\mu(\{A \in \Omega \,|\, X \cup Y \cup Z \subseteq A\}) \leq \frac{c_1 \log^{c} n}{m^{\gamma x}(n - m)^{\gamma t}} \prod_{i=1}^{x} \frac{1}{a_i^{\gamma}}.$$

This does not depend on the partition of X into $X = U \cup V \cup W$, and so

$$\sum_{U,V,W}^{(1)} \mu(\{A \in \Omega \mid X \cup Y \cup Z \subseteq A\}) \leq \frac{c_2 \log^c n}{m^{\gamma x}(n-m)^{\gamma t}} \prod_{i=1}^{x} \frac{1}{a_i^\gamma}$$

Then

$$\begin{aligned}\sum_{X}^{(2)} \sum_{U,V,W}^{(1)} \mu(\{A \in \Omega \mid X \cup Y \cup Z \subseteq A\}) &\leq \frac{c_2 \log^c n}{m^{\gamma x}(n-m)^{\gamma t}} \sum_{X}^{(2)} \prod_{i=1}^{x} \frac{1}{a_i^\gamma} \\ &\leq \frac{c_2 \log^c n}{m^{\gamma x}(n-m)^{\gamma t}} \left(\sum_{k=1}^{[(m-1)/2]} \frac{1}{k^\gamma} \right)^x \\ &\leq \frac{c_3 \log^c n}{m^{(2\gamma-1)x}(n-m)^{\gamma t}} \\ &\leq \frac{c_3 \log^c n}{m^{(2\gamma-1)t}(n-m)^{\gamma t}}\end{aligned}$$

since $\gamma \leq \frac{1}{2}$ and $x \leq t$. There are only a finite number of partitions of t in the form $t = u + 2v + w$, and so

$$\begin{aligned}\mu_t(m,n) &= \sum_{t}^{(3)} \sum_{X}^{(2)} \sum_{U,V,W}^{(1)} \mu(\{A \in \Omega \mid X \cup Y \cup Z \subseteq A\}) \\ &\leq \frac{c_4 \log^c n}{m^{(2\gamma-1)t}(n-m)^{\gamma t}}.\end{aligned}$$

If $2^k m \leq n < 2^{k+1} m$, then $n - m \geq (2^k - 1)m$ and $\log^c n \leq c''(k \log m)^{c'}$. Thus,

$$\mu_t(m,n) \leq \frac{c_5 k^{c'} (\log m)^{c'}}{m^{(3\gamma-1)t}(2^k-1)^{\gamma t}}.$$

Finally,

$$\begin{aligned}\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{n=2^k m}^{2^{k+1}m-1} \mu_t(m,n) &\leq \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{c_5 k^{c'} (\log m)^{c'} 2^k m}{m^{(3\gamma-1)t}(2^k-1)^{\gamma t}} \\ &= c_5 \sum_{m=1}^{\infty} \frac{(\log m)^{c'}}{m^{(3\gamma-1)t-1}} \sum_{k=1}^{\infty} \frac{k^{c'} 2^k}{(2^k-1)^{\gamma t}} \\ &< \infty.\end{aligned}$$

Both infinite series converge since $\gamma \leq \frac{1}{2}$ and $t > 2/(3\gamma - 1)$. This completes the proof.

THEOREM 2. *Let Ω be the space of all strictly increasing sequences of positive integers with the probability measure μ defined by a sequence $p(n)$ satisfying* (1) *and* (2). *For almost all $A \in \Omega$ and for all but finitely many pairs (m, n) of* *positive integers with* $m < n < 2m$,

$$|S_A(m) \cap S_A(n)| \leq 2/(3\gamma - 1).$$

Proof. Let $t > 2/(3\gamma - 1)$ and $m < n < 2m$. Define $\mu_t(m,n)$ by (12). We shall prove that

$$\sum_{m=1}^{\infty} \sum_{n=m+1}^{2m-1} \mu_t(m,n) < \infty.$$

Then the theorem follows from the Borel–Cantelli lemma.

The argument is similar to that of Theorem 1. We use formula (13) to estimate $\mu_t(m,n)$. However, since $n < 2m$, it is possible that $(X \cup Y) \cap Z \neq \varnothing$. Let us *assume* that $(X \cup Y) \cap Z = \varnothing$. Then

$$\begin{aligned}
&\mu(\{A \in \Omega \mid X \cup Y \cup Z \subseteq A\}) \\
&\quad \leq \prod_{i=1}^{x} p(a_i) \prod_{i=1}^{x} p(m - a_i) \prod_{i=1}^{u+v} p(n - a_i) \prod_{i=u+1}^{x} p(n - m + a_i) \\
&\quad \leq (\alpha \log^\beta 2m)^{2x+t} \prod_{i=1}^{x} \frac{1}{a_i^\gamma} \prod_{i=1}^{x} \frac{1}{(m - a_i)^\gamma} \\
&\qquad \times \prod_{i=1}^{u+v} \frac{1}{(n - a_i)^\gamma} \prod_{i=u+1}^{x} \frac{1}{(n - m + a_i)^\gamma}.
\end{aligned}$$

Using the inequalities

$$\begin{aligned}
m - a_i &> m/2 && \text{for} \quad i = 1, \ldots, x, \\
n - a_i &> m/2 && \text{for} \quad i = u+1, \ldots, u+v, \\
n - a_i &> a_i && \text{for} \quad i = 1, \ldots, u \\
n - m + a_i &> a_i && \text{for} \quad i = u+1, \ldots, x,
\end{aligned}$$

we obtain

$$\mu(\{A \in \Omega \mid X \cup Y \cup Z \subseteq A\}) \leq \frac{c_1 \log^c m}{m^{\gamma t}} \prod_{i=1}^{x} \frac{1}{a_i^{2\gamma}}.$$

Therefore, by (13),

$$\begin{aligned}
\mu_t(m,n) &= \sum_{t}^{(3)} \sum_{X}^{(2)} \sum_{U,V,W}^{(1)} \mu(\{A \in \Omega \mid X \cup Y \cup Z \subseteq A\}) \\
&\leq \frac{c_2 \log^c m}{m^{\gamma t}} \left(\sum_{k=1}^{[(m-1)/2]} \frac{1}{k^{2\gamma}} \right)^t \\
&\leq \frac{c_3 \log^c m}{m^{(3\gamma - 1)t}}.
\end{aligned}$$

Finally,

$$\sum_{m=1}^{\infty}\sum_{n=m+1}^{2m-1}\mu_t(m,n) \le c_3 \sum_{m=1}^{\infty}\frac{\log^c m}{m^{(3\gamma-1)t-1}} < \infty$$

since $t > 2/(3\gamma - 1)$. The proof in the case $(X \cup Y) \cap Z \neq \varnothing$ is similar.

Theorem 1 and Theorem 2 are useful in the study of extremal sequences in additive number theory. For example, they provide a proof of the existence of minimal bases. An asymptotic basis A of order 2 is *minimal* if no proper subset of A is a basis. This means that for every $a \in A$ there are infinitely many positive integers n such that $n \notin 2(A\backslash\{a\})$. It is not true that every basis contains a subset that is a minimal basis [4]. However, the following result gives a simple criterion for a basis to contain a minimal basis.

THEOREM 3. *Let A be a strictly increasing sequence of positive integers such that*

(i) $\lim_{n\to\infty} r_A(n) = \infty$,
(ii) $|S_A(m) \cap S_A(n)|$ *is bounded for all* $m < n$.

Then A contains a minimal asymptotic basis of order 2.

Proof. Let $|S_A(m) \cap S_A(n)| \le d - 1$ for all $m < n$. Define

$$P_A(n) = \{a \in A \mid n - a \in A \text{ and } a \ge n/2\}.$$

Then $P_A(n) \subseteq S_A(n) \cup \{n/2\}$. Fix n_1 so that $r_A(n) > d$ for all $n \ge n_1$. Choose $a_1^* \in A$. Let $a_1 \in A$ satisfy $a_1 > \max(a_1^*, 2n_1)$. Let $m_1 = a_1^* + a_1$. Then $a_1 \in P_A(m_1)$ and $a_1^* \notin P_A(m_1)$. Define

$$A_1 = A\backslash(P_A(m_1)\backslash\{a_1\}).$$

Then $a_1, a_1^* \in A_1$ and so $m_1 = a_1^* + a_1 \in 2A_1$.

Let $n \ge n_1$ and $n \ne m_1$. Since

$$A\backslash A_1 \subseteq P_A(m_1) \subseteq S_A(m_1) \cup \{m_1/2\},$$

it follows that

$$P_A(m_1) \cap S_A(n) \subseteq (S_A(m_1) \cap S_A(n)) \cup \{m_1/2\},$$

and so

$$\begin{aligned} r_{A_1}(n) &\ge r_A(n) - |(A\backslash A_1) \cap S_A(n)| \\ &\ge r_A(n) - |P_A(m_1) \cap S_A(n)| \\ &\ge r_A(n) - d \\ &\ge 1. \end{aligned}$$

Therefore, $n \in 2A_1$ for all $n \geq n_1$, and so A_1 is a basis. Moreover, $m_1 = a_1^* + a_1$ is the unique representation of m_1 as a sum of two elements of A_1.

Let $k \geq 2$. Suppose we have constructed integers a_i, a_i^*, m_i, n_i for $i = 1, \ldots, k-1$ and sets $A_1, \ldots, A_{k-1}$ with the following properties:

(i) $2n_1 < m_1 < 2n_2 < m_2 < \cdots < 2n_{k-1} < m_{k-1}$;

(ii) $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{k-1}$;

(iii) $A_{i-1} \backslash A_i \subseteq [m_i/2, m_i]$;

(iv) $a_i, a_i^* \in A_i$ for $i = 1, \ldots, k-1$;

(v) $m_i = a_i^* + a_i$ for $i = 1, \ldots, k-1$, and this is the unique representation of m_i as a sum of two elements of A_i;

(vi) if $n \geq n_1$, then $n \in 2A_{k-1}$.

We now construct a_k, a_k^*, m_k, n_k, and A_k.

Choose $n_k > m_{k-1}$ such that $r_A(n) > d + m_{k-1}$ for all $n \geq n_k$. Choose $a_k^* \in A_{k-1}$ with $a_k^* < m_{k-1}$. Choose $a_k \in A_{k-1}$ such that $a_k > 2n_k > a_k^*$. Let $m_k = a_k^* + a_k$. Define

$$A_k = A_{k-1} \backslash (P_{A_{k-1}}(m_k) \backslash \{a_k\}).$$

Then a_k, a_k^*, m_k, n_k, and A_k satisfy conditions (i)–(v).

We must show that $n \in 2A_k$ for all $n \geq n_1$. Since $A_{k-1} \backslash A_k \subseteq [m_k/2, m_k] \subseteq [n_k, m_k]$, it follows from (vi) that $n \in 2A_k$ if $n_1 \leq n < n_k$. Let $n \geq n_k$, $n \neq m_k$. Since $A \backslash A_{k-1} \subseteq [1, m_{k-1}]$, it follows that

$$\begin{aligned} A \backslash A_k &\subseteq [1, m_{k-1}] \cup P_{A_{k-1}}(m_k) \\ &\subseteq [1, m_{k-1}] \cup S_A(m_k) \cup \{m_k/2\}. \end{aligned}$$

Therefore,

$$\begin{aligned} r_{A_k}(n) &\geq r_A(n) - |(A \backslash A_k) \cap S_A(n)| \\ &\geq r_A(n) - m_{k-1} - 1 - |S_A(m_k) \cap S_A(n)| \\ &\geq r_A(n) - m_{k-1} - d \\ &\geq 1, \end{aligned}$$

and so A_k satisfies (vi).

Continuing inductively, we obtain infinite sequences a_k, a_k^*, m_k, n_k, and A_k satisfying properties (i)–(vi). Define

$$A^* = \bigcap_{k=1}^{\infty} A_k.$$

If $n \geq n_1$, then $n \in 2A^*$ and so A^* is an asymptotic basis of order 2. Moreover, $m_k = a_k^* + a_k$ is the unique representation of m_k as the sum of two elements of A^*, and so $m_k \notin 2(A^* \backslash \{a_k^*\})$.

Here is the key idea for the construction of a minimal basis. In the kth step of the induction, we could choose arbitrarily $a_k^* \in A_{k-1}$ such that $a_k^* < m_{k-1}$. We make these choices in such a way that if $a^* \in A^*$, then $a^* = a_k^*$ for infinitely many k. Then for every $a^* \in A$ there are infinitely many integers m_k such that $m_k \notin 2(A^* \setminus \{a^*\})$. Thus, A^* is a minimal basis contained in A. This completes the proof of Theorem 3.

Let Ω be the probability space of sequences of positive integers defined by $p(1) = \frac{1}{2}$ and $p(n) = \alpha((\log n)/n)^{1/2}$ for $n \geq 2$. By the theorem of Erdös and Rényi [1], there exists $c > 0$ such that $r_A(n) > c \log n$ for almost all $A \in \Omega$ and all n sufficiently large. Theorems 1 and 2 imply that $|S_A(m) \cap S_A(n)| \leq 4$ for almost all $A \in \Omega$ and all but finitely many pairs (m, n) with $m < n$. It follows from Theorem 3 that the sequence A contains a minimal basis for almost all $A \in \Omega$.

4. Open Problems

We do not know whether it is possible to improve the right-hand side of the inequality

$$|S_A(m) \cap S_A(n)| \leq 2/(3\gamma - 1)$$

in Theorems 1 and 2. In particular, with $\gamma = \frac{1}{2}$ and $p(n) = \alpha((\log n)/n)^{1/2}$ for $n \geq 2$, we do not know whether $|S_A(m) \cap S_A(n)| \leq 3$ for almost all $A \in \Omega$ and all but finitely many pairs $m < n$. We can prove that for $k \geq 2$ and almost all $A \in \Omega$ there exist infinitely many pairwise disjoint k-tuples $m_1 < \cdots < m_k$ such that

$$|S_A(m_1) \cap S_A(m_2) \cap \cdots \cap S_A(m_k)| \geq 2.$$

We do not know whether condition (ii) in Theorem 3 is necessary. It is possible that there exists a sequence A of positive integers that does not contain a minimal basis but does satisfy the condition $\lim_{n\to\infty} r_A(n) = \infty$.

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