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SOME REMARKS ON SUBGROUPS OF REAL NUMBERS

BY

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A problem of Stanisław Hartman states: Is there a group of real numbers which is of measure 0 and of second category? Also, is there one of first category and not of measure 0?

Using the continuum hypothesis I will prove that such groups exist. Without much extra trouble one can show that there are fields of real numbers with the same properties.

In some related problems the change from group to ring or field causes a great deal of difficulty. A theorem of Volkmann and myself [5] states that for every $0 \leqslant \alpha \leqslant 1$ there is a group of real numbers of Hausdorff dimension α. All our efforts so far failed in proving the existence of a ring or field of Hausdorff dimension α.

Before giving the simple proofs I state a few related results which I proved during my long life and also formulate a few problems.

Kakutani and I proved [4] that the real line is the union of $\aleph_0$ rationally independent sets if and only if $2^{\aleph_0} = \aleph_1$.

I then asked: Can one decompose the n-dimensional space as the union of $\aleph_0$ sets E_k $(k = 1, 2, \ldots)$ so that all the distances in E_k $(k = 1, 2, \ldots)$ occur only once? Davies [1] proved this for $n = 2$. The cases $n \geqslant 3$ are open.

Let $\{a_\alpha\}$ be a Hamel basis. I proved [2] that if H_k is the set of reals which have exactly k non-zero terms in their canonical representation, then for every k there is a Hamel basis for which H_k is non-measurable and H_i $(i < k)$ is of measure 0. Clearly, for every Hamel basis there is a k such that H_i $(i \geqslant k)$ is non-measurable.

It is easy to see that for every Hamel basis the set of reals for which all summands in the canonical representation have integer coefficients is non-measurable. But if $2^{\aleph_0} = \aleph_1$, there is a Hamel basis such that the set of real numbers all coefficients of which in the canonical representation are positive is of measure 0 (see [3]).

THEOREM. *If $2^{\aleph_0} = \aleph_1$, then there are groups of real numbers which are*

(a) *of measure* 0 *and of second category;*

(b) *of first category and not of measure* 0.

Proof. Let $A_0, \ldots, A_\xi, \ldots$ $(\xi < \omega_1)$ be the sequence of all sets in R which are F_σ and of first category (G_δ and of measure 0). We construct a sequence of groups $G_\xi \subseteq R$ $(\xi < \omega_1)$ such that

$$\overline{\overline{G}}_\xi = \aleph_0 \quad \text{for all } \xi < \omega_1, \tag{1}$$

$$G_\xi \subset G_\tau \quad \text{and} \quad G_\xi \neq G_\tau \quad \text{for } \xi < \tau < \omega_1, \tag{2}$$

$$G_\xi \cap \bigcup_{\chi<\xi} A_\chi = G_\tau \cap \bigcup_{\chi<\xi} A_\chi \quad \text{for } \xi < \tau < \omega_1. \tag{3}$$

If this is done, the group

$$G = \bigcup_{\xi<\omega_1} G_\xi$$

has the required properties: it is of measure 0 because its intersection with a certain A_ξ of full measure is countable, it is not of first category because it is not contained in any of the A_ξ. (It is of first category because its intersection with a certain residual A_ξ is countable, it is not of measure 0 because it is not contained in any of the A_ξ.)

We construct G_ξ $(\xi < \omega_1)$ by induction. Suppose that G_ξ $(\xi < \tau)$ is ready. G_τ will be the group generated by $\bigcup_{\xi<\tau} G_\xi$ and a number $x \in R \setminus \bigcup_{\xi<\tau} G_\xi$ chosen as follows: x does not belong to any of the sets

$$\{y\colon\ ny + g \in A_\xi\}, \tag{4}$$

where $\xi < \tau$, n is any integer not equal to 0 and $g \in \bigcup_{\xi<\tau} G_\xi$. Clearly, there are countably many sets (4) and each is of first category (of measure 0), hence there exists an x as required.

It is clear that G_τ satisfies (1), (2), and (3).

REFERENCES

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[3] — *On some properties of Hamel bases*, Colloquium Mathematicum 10 (1963), p. 267-269.

[4] — and J. Kakutani, *On non-denumerable graphs*, Bulletin of the American Mathematical Society 49 (1943), p. 457-461.

[5] P. Erdös und B. Volkmann, *Additive Gruppen mit vorgegebener Hausdorffscher Dimension*, Journal für die reine und angewandte Mathematik 221 (1966), p. 203-208.

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