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ON THE GROWTH OF SOME ADDITIVE FUNCTIONS ON SMALL INTERVALS

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1. The letters $c, c_1, c_2, \dots$ denote suitable, $\varepsilon, \varepsilon_1, \varepsilon_2, \dots, \delta$ small positive constants. $\varepsilon_1, \varepsilon_2, \dots$ will depend on ε. p_n denotes the n^{th} prime number, $p, q, q_1, q_2, \dots$ are primes. $\sum_p$ denotes a summation over primes indicated. $\pi(x) = \sum_{p \leqq x} 1$. $\omega(n)$ denotes the number of distinct prime factors of n. (a, b) and $[a, b]$ denote the greatest common divisor and the least common multiple of a and b, resp. $[x]$ denotes the integer part of x. For the sake of brevity we shall write $x_{i+1} = \log x_i$ $(i=0, 1, 2)$, $x_0 = x$.

Let

$$O_k(n) = \max_{j=1,\dots,k} \omega(n+j), \quad o_k(n) = \min_{j=1,\dots,k} \omega(n+j). \tag{1.1}$$

One of us (see [1]) proved the following assertions. For every $\varepsilon > 0$, apart from a set of n's having zero density, the inequalities

$$O_k(n) \leqq (1+\varepsilon)\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n, \quad o_k(n) \geqq (1-\varepsilon)\bar{\varrho}\left(\frac{\log k}{\log\log n}\right)\log\log n$$

hold for every $k = 1, 2, \dots$. Here $\varrho(u)$ $(u \geqq 0)$ is defined as the inverse function of $\psi(r) = r\log\frac{z}{e} + 1$ defined in $z \geqq 1$, and $\bar{\varrho}(n)$ $(n \geqq 0)$ is the inverse function of the same $\psi(r)$ defined in $0 < z \leqq 1$. In the same paper it was conjectured that

$$O_k(n) \geqq (1-\varepsilon)\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n, \tag{1.2}$$

and

$$o_k(n) \leqq (1+\varepsilon)\bar{\varrho}\left(\frac{\log k}{\log\log n}\right)\log\log n,$$

for every $k \geqq 1$ and for almost all n. The last conjecture is false, since for $k = \log n$, $o_k(n) = 0$ would follow, which is impossible. Instead of it we state

$$o_k(n) \leqq \left\{\bar{\varrho}\left(\frac{\log k}{\log\log n}\right) + \varepsilon\right\}\log\log n, \tag{1.3}$$

where $\bar{\varrho}(u) = 0$ or $u \geqq 1$.

We shall prove

THEOREM 1. *For every $\varepsilon > 0$ the inequalities* (1.2), (1.3) *hold for every $k \geqq 1$, apart from a set of n's having zero density.*

Let $g(n)$ be a non-negative strongly additive function, i.e. $g(p^\alpha)=g(p)$ for every prime p. Let

$$f_k(n) = \max_{j=1,\dots,k} g(n+j). \tag{1.4}$$

It is obvious that $f_k(n) \geqq f_k(0)$. We are interested in the conditions which imply that

$$f_k(n) \leqq (1+\varepsilon) f_k(0) \tag{1.5}$$

holds for every $k > k_0$, apart from a set of n's having upper density at most $\delta(\varepsilon, k_0)$, where $\delta(\varepsilon, k_0) \to 0$ as $k_0 \to \infty$.

This question was considered for some special functions in [2].

Let

$$g^+(p) = \begin{cases} g(p), & \text{if } g(p) \leqq 1, \\ 1, & \text{if } g(p) > 1, \end{cases}$$

and $g^+(n)$ is defined as a strongly additive function generated by the values $g^+(p)$. By using the wellknown Turán—Kubilius inequality

$$\sum_{n \leqq x} (g^+(n) - A_x)^2 \leqq c \times B_x \quad (\leqq c \times A_x)$$

$$A_x = \sum_{p \leqq x} \frac{g^+(p)}{p}, \quad B_x = \sum_{p \leqq x} \frac{g^{+2}(p)}{p} \quad (\leqq A_x),$$

and that $g(n) \geqq g^+(n)$, we immediately have that the convergence of

$$\sum \frac{g^+(p)}{p}$$

is a necessary condition for the truth of (1.5).

We are unable to decide if

$$\sum \frac{g(p)}{p} < \infty \tag{1.6}$$

is necessary for (1.5).*

Assume that $g(p)$ tends to zero monotonically as $p \to \infty$. We shall prove that (1.6) is not sufficient for (1.5). This disproves the conjecture stated in [2], namely that from the convergence of the series $\sum \frac{g^+(p)}{p}$, $\sum_{g(p)>1} \frac{1}{p} > 1$ (1.5) would follow. Finally, assuming some regularity conditions on

$$A(y) = \sum_{p \leqq y} g(p)$$

we shall show that (1.5) holds.

Let $t(x)$ be a real valued monotonically decreasing function defined for $x \geqq 1$. Let

$$A(y) = \sum_{p \leqq y} t(p), \tag{1.7}$$

* REMARK. We decided this question affirmatively. We shall publish this in a forthcoming paper in this journal.

and suppose that

$$\sum_p \frac{t(p)}{p} < \infty, \tag{1.8}$$

and that for every positive constant δ

$$\lim_{y\to\infty} \frac{A(y)}{yt(\exp(\exp(y^\delta)))} = \infty.$$

Let $g(n)$ be the strongly additive function defined for primes as $g(p)=t(p)$.

THEOREM 2. *Assume that the conditions* (1.7), (1.8) *hold. Let* ε *be an arbitrary positive constant. Then for every integer* k_0 *the inequality*

$$f_k(n) < (1+\varepsilon) f_k(0)$$

holds for every $k \geqq k_0$ *and for all but* $\delta(k_0, \varepsilon)x$ *integers* n *in* $[1, x]$. *Here* $\delta(k_0, \varepsilon) \to 0$ $(k_0 \to \infty)$.

We shall prove these assertions in the following sections.

Now we make the following remark. In [3], IVÁNYI and KÁTAI proved the existence of a completely additive $f(n)$ not identically zero for which $f(n)=A_j$, $n\in[N_j, N_j+\tau(N_j)]$ on a suitable set $N_1<N_2<\dots$ of integers, where $\tau(N)= \exp\left(c\sqrt{(\log N)(\log\log\log N)}\right)$, A_j are arbitrary complex or real values.

Now we prove the following

THEOREM 3. *Let* $\varepsilon>0$ *and* $x>x_0(\varepsilon)$. *Then there exists a completely additive function* $f(n)$ *for which*

$$f(n) = 0 \quad in \quad [N+1, N+\lambda(x)],$$

where $\frac{x}{2} \leqq N \leqq x$ *and*

$$\lambda(x) = \left[\exp\left(\left(\frac{1}{2}-\varepsilon\right)\frac{(\log x)(\log\log\log x)}{\log\log x}\right)\right],$$

and which takes on a non-zero value in $[1, \sqrt{x}]$.

REMARK. Unfortunately we can not prove that there is an $f(n)$ with infinitely many such intervals.

PROOF. Denote by $N(x, y)$ the number of integers $n \leqq x$ all prime factors of which are not greater than y. By a theorem of RANKIN [4]

$$N(x, y) < x\exp\left(-\frac{\log\log\log y}{\log y}\log x + \log\log y + O\left(\frac{\log\log y}{\log\log\log y}\right)\right). \tag{1.9}$$

Let $k=\lambda(x)$, x large. (1.9) implies

$$N(x, k) < \left[\frac{x}{2k}\right]\pi(k).$$

Thus it is easy to see that there is an interval $[N+1, N+k]$ in $\frac{x}{2} \leqq N < N+k \leqq x$, for which the number of integers all prime factors of which do not exceed k is smaller than $\pi(k)$. Let $n=A(n)B(n)$, where $A(n)$ is composed of the prime factors $\leqq k$ of n. Let $n+l_i$ $(i=1, \ldots, h)$, $h<\pi(k)$ be the n's in $[N+1, N+k]$ for which $B(n+l_i)=1$.

The additivity leads to the following linear system of equations:

$$f(A(n+l_j))=0 \quad (j=1, \ldots, h), \tag{1.10}$$

$$f(B(n+r))=-f(A(n+r)) \quad (r \neq l_j(j=1, \ldots, h)), \tag{1.11}$$

where the indeterminates are the values $f(p)$ for primes p contained in $(N+1), \ldots$ $\ldots, (N+k)$. (1.9) is a homogeneous system, the number h of equations is smaller than $\pi(k)$, therefore we can choose values $f(p_1), \ldots, f(p_{\pi(k)})$ non-trivially such that (1.10) hold. This holds in the case $h=0$, too. To finish the proof we need to take into account only that $B(n+r)$ $(r\neq l_j, j=1, \ldots, k)$ are mutually coprime, so we can solve (1.11). This completes the proof of Theorem 3.

2. Lemmas. Let k be an integer, $\mathscr{P}$ be a finite set of primes greater than k. Let $\mathscr{T}_r$ denote the set of integers of the form $t_r=q_1q_2\ldots q_r$, $q_i\in\mathscr{P}$, $q_i\neq q_j$ $(i\neq j)$,

$$P=\sum_{p\in\mathscr{P}} 1/p, \quad T_r=\sum_{t_r\in\mathscr{T}_r} 1/t_r, \tag{2.1}$$

$$a=\sum_{p\in\mathscr{P}}\frac{1}{p^2}. \tag{2.2}$$

Let Π_r be the number of elements of $\mathscr{T}_r$.

LEMMA 1. *For every $r\geqq 2$ we have*

$$\frac{P^r}{r!}-\frac{a}{2}\frac{P^{r-2}}{(r-2)!}\leqq T_r\leqq\frac{P^r}{r!}. \tag{2.3}$$

PROOF. The right hand side of (2.3) is obvious. We prove the left hand side by using induction. The assertion holds for $r=2$, since

$$T_2=\frac{1}{2}(P^2-a).$$

Observing that

$$T_rP\leqq T_{r+1}(r+1)+\sum_{p\in\mathscr{P}}\frac{1}{p^2}\Big\{\sum_{(t_{r-1},p)=1}\frac{1}{t_{r-1}}\Big\}\leqq T_{r+1}(r+1)+aT_{r-1},$$

we get

$$T_{r+1}\geqq\frac{T_rP}{r+1}-\frac{a}{r+1}T_{r-1},$$

and by the induction hypothesis

$$T_{r+1}\geqq\Big\{\frac{P^r}{r!}-\frac{a}{2}\frac{P^{r-2}}{(r-2)!}\Big\}\frac{P}{r+1}-\frac{a}{r+1}\frac{P^{r-1}}{(r-1)!}=\frac{P^{r+1}}{(r+1)!}-\frac{a}{2}\frac{P^{r-1}}{(r-1)!}.$$

By this Lemma 1 is proved.

We shall use Brun's sieve in the form of Theorem 2.5 in [5], or in the simpler form of [6], Theorem 6.2. Namely we shall use the following result, which we state now as

LEMMA 2. *Let $a_1, a_2, \ldots$ be positive integers, $\mathcal{R}$ a finite set of primes, all of them smaller than z. Let*

$$\eta(y, d) = \left| \sum_{\substack{a_\nu \equiv o(d) \\ a_\nu \leqq y}} 1 - \frac{\gamma(d)}{d} y \right|,$$

where $\gamma(d)$ is a multiplicative function on the set of square free numbers all prime factors of which are in $\mathcal{R}$. Suppose that $\eta(y, d) \leqq \gamma(d)$ for all such d, and $\gamma(p) = O(1)$, $\gamma(p) \leqq p-1$ for all $p \in \mathcal{R}$. Putting $R = \prod_{p \in \mathcal{P}} p$, for $y \geqq r$ we get

$$\sum_{\substack{a_\nu \leqq y \\ (a_\nu, R)=1}} 1 = y \prod_{p \in \mathcal{R}} \left(1 - \frac{\gamma(p)}{p}\right) \left\{1 + O\left(\exp\left(-\frac{1}{2} \frac{\log y}{\log z}\right)\right)\right\}. \tag{2.4}$$

Let now $\mathcal{P}$ be the set of all primes in (k, r), where $z < x^{1/4r}$. Let $\mathcal{A}$ be the set of integers $n = t_r b$, where $t_r \in \mathcal{T}_r$, $(b, \prod_{p \in \mathcal{P}} p) = 1$. Let

$$V(n) = \begin{cases} 1, & \text{if } n \in \mathcal{A}, \\ 0, & \text{if } n \notin \mathcal{A}, \end{cases}$$

and put

$$\sum\nolimits^{(0)} = \sum_{n \leqq x} V(n), \quad \sum\nolimits^{(h)} = \sum_{n+h \leqq x} V(n) V(n+h) \quad (h = 1, \ldots, k). \tag{2.5}$$

Let

$$\Gamma_1 = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right), \quad \Gamma_2 = \prod_{p \in \mathcal{P}} \left(1 - \frac{2}{p}\right), \tag{2.6}$$

and $\lambda(n)$ a multiplicative function on the square free integers defined for primes p by $\lambda(p) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)^{-1}$.

For the computation of $\sum^{(0)}, \sum^{(h)}$ we shall use the previous lemma. Let $N(y|\mathcal{P})$ be the number of $b \leqq y$, which have no prime factors in $\mathcal{P}$. By (2.4),

$$N(y|\mathcal{P}) = y\Gamma_1 \left\{1 + O\left(\exp\left(-\frac{2r \log y}{\log x}\right)\right)\right\},$$

since $x/t_r \geqq x^{3/4}$. Consequently

$$\sum\nolimits^{(0)} = \sum_{t_r \in \mathcal{T}_r} N\left(\frac{x}{t_r} | \mathcal{P}\right) = T_r \Gamma_1 \times (1 + O(e^{-r})). \tag{2.7}$$

Consider now $\sum^{(h)}$ $(h \geqq 1)$. First we count the integers n, $n = t_r^{(1)} b_1$, $n + h = t_r^{(2)} b_2 \leqq x$ with fixed $t_r^{(1)}, t_r^{(2)} \in \mathcal{T}_r$. There is a solution only if $(t_r^{(1)}, t_r^{(2)}) = 1$. The solutions b_1, b_2 of $t_r^{(2)} b_2 - t_r^{(1)} b_1 = h$ are in the progressions $b_2 = b_2^{(0)} + s t_r^{(1)}$,

$b_1 = b_1^{(0)} + st_r^{(1)}$ ($s=0, 1, 2, \ldots$). Sieving those elements $b_1 b_2$ which have prime factors in $\mathcal{P}$, we get that $\gamma(p)=2$ if $p \nmid t_r^{(1)} t_r^{(2)}$, and $\gamma(p)=1$, if $p \mid t_r^{(1)} t_r^{(2)}$. Thus by Lemma 2,

(2.8)
$$\sum{}^{(h)} = x\Gamma_2 \big(1 + O(\bar{e}^r)\big) A,$$

$$A = \sum_{(t_r^{(1)}, t_r^{(2)})=1} \frac{\lambda(t_r^{(1)} t_r^{(2)})}{t_r^{(1)} t_r^{(2)}}.$$

Since $t_r^{(1)} t_r^{(2)} = t_{2r}$ has $\binom{2r}{r}$ solutions for fixed t_{2r} we have

$$A = \binom{2r}{r} \sum \frac{\lambda(t_{2r})}{t_{2r}}.$$

Let $h(d)$ be the Moebius transform of $\lambda(d)$. Then $h(p) = \dfrac{1}{p-2}$, $h(d)$ is multiplicative, and we have

$$T_{2r} \leqq \sum \frac{\lambda(t_{2r})}{t_{2r}} \leqq T_{2r} + \sum_{\nu=1}^{2r} \left\{ \sum_{\delta \in \mathcal{T}_\nu} \frac{h(\delta)}{\delta} \right\} T_{2r-\nu}.$$

Taking into account that

$$\sum_{\delta \in \mathcal{T}_\nu} \frac{h(\delta)}{\delta} \leqq \frac{1}{\nu!} \left\{ \sum_{p>k} \frac{1}{p(p-2)} \right\}^\nu \leqq \frac{1}{\nu!} \left(\frac{c}{k \log k} \right)^\nu,$$

from Lemma 1 we get

$$\sum \frac{\lambda(t_{2r})}{t_{2r}} \leqq \frac{P^{2r}}{(2r)!} \exp\left(\frac{2rc}{Pk \log k} \right).$$

Furthermore Lemma 1 implies that

$$T_{2r} \geqq \left(1 - \frac{4ar}{P} \right) \frac{P^{2r}}{(2r)!},$$

and so

$$A = \left(\frac{P^r}{r!} \right)^2 \left(1 + O\left(\frac{r}{Pk \log k} \right) \right),$$

if

(2.9)
$$\frac{r}{Pk \log k} = O(1).$$

We have

$$\log \Gamma_2 = 2 \log \Gamma_1 + O(a), \quad \log \Gamma_1 = -P + O(a),$$

whence

$$\Gamma_1 = e^{-P}\big(1 + O(a)\big), \quad \Gamma_2 = e^{-2P}\big(1 + O(a)\big).$$

Consequently

(2.10)
$$\sum{}^{(0)} = xe^{-P} \frac{P^r}{r!^2} \left(1 + O(e^{-r}) + O\left(\left(\frac{r}{P} + 1 \right) \frac{1}{k \log k} \right) \right),$$

(2.11)
$$\sum{}^{(h)} = xe^{-2P} \frac{P^{2r}}{r!} \left(1 + O(e^{-r}) + O\left(\left(\frac{r}{P} + 1 \right) \frac{1}{k \log k} \right) \right),$$

if (2.5) holds.

Let

$$F_k(n) = \sum_{i=1}^{k} V(n+i), \quad \Lambda = ke^{-p}\frac{P^r}{r!}, \tag{2.12}$$

$$E = \sum_{n \leqq x} (F_k(n) - \Lambda)^2. \tag{2.13}$$

We have

$$E = \sum_{n \leqq x} F_k^2(n) - 2\Lambda \sum_{n \leqq x} F_k(n) + \Lambda^2 x,$$

and observe that

$$\sum_{n \leqq x} F_k(n) = k \sum^{(0)} + O(k^2),$$

$$\sum_{n \leqq x} F_k^2(n) = k \sum^{(0)} + \sum_{h=1}^{k} 2(k-h) \sum^{(h)} + O(k^3).$$

Collecting our results we get

LEMMA 3. *If* (2.9) *holds, then*

$$E = O\left(x(\Lambda^2 + \Lambda)\left(e^{-r} + \frac{r+P}{Pk \log k}\right) + k^3 + k^2 \Lambda\right). \tag{2.14}$$

Let now $\mathscr{P}$ be an arbitrary set of primes, $P = \sum_{p \in \mathscr{P}} 1/p$,

$$\omega(n|\mathscr{P}) = \sum_{\substack{p|n \\ p \in \mathscr{P}}} 1, \tag{2.15}$$

$$O_k(n) = \max_{j=1,\dots,k} \omega(n+j|\mathscr{P}), \quad o_k(n) = \min_{j=1,\dots,k} \omega(n+j|\mathscr{P}). \tag{2.16}$$

Let $D_k(x, L|\mathscr{P})$ be the number of $n \leqq x$ for which $O_k(n|\mathscr{P}) \geqq L$. It is obvious that

$$D_k(x, L|\mathscr{P}) \leqq z^{-L} \sum_{n \leqq x} z^{O_k(n|\mathscr{P})} \leqq z^{-L} k \sum_{n \leqq x+k} z^{\omega(n|\mathscr{P})},$$

for $z \geqq 1$. Observing that

$$\sum_{n \leqq x+k} z^{\omega(n|\mathscr{P})} \leqq (x+k) \prod_{p \in \mathscr{P}} \left(1 + \frac{z-1}{p}\right) < (x+k) \exp(zP),$$

by substituting $z = L/p$, we get immediately

LEMMA 4. *If* $1 \leqq k \leqq x$, $L \geqq P$, *then*

$$D_k(x, L|\mathscr{P}) \leqq 2x \exp\left(\log k - L \log \frac{L}{Pe}\right). \tag{2.17}$$

3. Proof of Theorem 1. First we prove (1.2). Let B be a suitable large constant depending on ε. First we shall prove (1.2) for

$$k \geqq \exp((\log \log n)^B). \tag{3.1}$$

Indeed, if we define t_k to be the largest integer l so that the product of the first l primes is smaller than k, then we get $O_k(n) \geqq O_k(0) = t_k$. From the prime number theorem we get

$$\log k \sim \sum_{j=1}^{t_k} \log p_j \sim p_{t_k} \sim t_k \log t_k,$$

whence

$$t_k \sim \frac{\log k}{\log\log k} \quad (k \to \infty).$$

Furthermore, as it is easy to show, $\varrho(u) \sim \frac{u}{\log u}$ $(u \to \infty)$, whence

$$\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n \geqq \left(1-\frac{\varepsilon}{2}\right)\frac{\log k}{\log\log k},$$

if B is large enough. Thus (1.2) holds if (3.1) satisfies.

Let B be fixed, x large, and put

$$\alpha = \frac{\log k}{x_2}. \tag{3.2}$$

Observing that $\varrho(\lambda) \sim 1+\sqrt{2\lambda}$ $(\lambda \sim 0)$, therefore by choosing ε_1 to satisfy $(1+2\sqrt{\varepsilon_1})\left(1-\frac{\varepsilon}{2}\right)<1$, we get $\left(1-\frac{\varepsilon}{2}\right)\varrho(\varepsilon_1)<1$. We can choose $\varepsilon_1 = \frac{\varepsilon^2}{16}$. By using Hardy—Ramanujan's wellknown theorem that $\omega(n) \sim \log\log n$ for almost all n, we get (1.2) in $0 \leqq \alpha \leqq \varepsilon_1$.

Assume that

$$\varepsilon_1 x_2 \leqq \log k \leqq x_2^B. \tag{3.3}$$

Let r be an integer for which

$$r = \Delta x_2 + O(1), \quad \Delta = (1-\varepsilon_2)\varrho(\alpha), \tag{3.4}$$

ε_2 being a small positive constant.

Let $\mathscr{P}$ be the set of primes in $(k, x^{1/4r})$ and $N_{k,r}(x)$ denote the number of $n \leqq x$ for which $O_k(n) < r$. For these numbers $F_k(n) = 0$, and by Lemma 4

$$N_{k,r}(x) \leqq \frac{E}{\Lambda^2} \leqq O\left(x\left(1+\frac{1}{\Lambda}\right)\left(e^{-r}+\frac{r+P}{Pk\log k}\right)+\frac{k^3+k^2\Lambda}{\Lambda^2}\right). \tag{3.5}$$

From (3.3), (3.4) we have

$$\alpha \leqq x_2^{B-1}, \quad \Delta \leqq c x_2^{B-1}, \quad \log r = O(x_3),$$

$$P = x_2 + O(x_3),$$

$$\frac{r+P}{Pk\log k} \ll \frac{(\Delta+1)x_2}{x_2 e^{\alpha x_2}\alpha x_2} = O(x_1^{-\alpha/2}).$$

By using Stirling formula,

$$\log\Lambda = \log k - P - r\log\frac{r}{Pe} + O(\log r) = (\alpha - \psi(\Delta))x_2 + O(x_3).$$

Since

$$\psi(\Delta) = (1-\varepsilon_2)\psi(\varrho) + \varepsilon_2 + (1-\varepsilon_2)\varrho\log(1-\varepsilon_2)$$

and $\psi(\varrho)=\alpha$, therefore by using that $\varrho(\lambda)\sim 1+\sqrt{2\lambda}$ $(\lambda\sim 0)$, we get $\alpha-\psi(\Delta)\geqq \geqq \varepsilon_2^2/2$, if $\alpha\geqq 4\varepsilon_2^2$, ε_2 being small. Choosing $\varepsilon_2\leqq\sqrt{2\varepsilon_1}$, we get that $\Lambda\geqq 1$ for all large x and for all α in (3.3).

Since $e^{-r}\ll e^{-\Delta x_2}$, we obtain that

$$N_{k,r}(x)\leqq c_2 x\{e^{-\Delta x_2}+e^{-\alpha x_2/2}\}+O(x^{1/2}). \tag{3.6}$$

Let now $\alpha_j=j\varepsilon_1$, $k_j=[e^{\alpha_j x_2}]$, $j=1,\dots,T$, and $T-1$ is the largest integer for which $\alpha_{T-1}\leqq x_2^{B-1}$. Thus $T=O\left(\frac{1}{\varepsilon_1}x_2^{B-1}\right)$, and from (3.6)

$$\sum_{i=1}^{T} N_{k_i,r}(x)\ll xe^{-\frac{\varepsilon_1}{3}x_2}. \tag{3.7}$$

Hence it follows that for all but $O\left(xx_1^{-\frac{\varepsilon_1}{3}}\right)$ integers n in $\left[\frac{x}{2},x\right]$

$$O_{k_i}(n)>\left(1-\frac{\varepsilon}{2}\right)\varrho\left(\frac{\log k_i}{x_2}\right)x_2 \quad (i=1,\dots,T). \tag{3.8}$$

Let $k\in[k_i,k_{i+1})$ and suppose that (3.8) holds for an n. Since $O_k(n)\geqq O_{k_i}(n)$ and $\varrho(\alpha)<(1+c_3\varepsilon_1)\varrho(\alpha_i)$, therefore

$$O_k(n)>\left(1-\frac{2\varepsilon}{3}\right)\varrho(\alpha)\log\log n.$$

Since $\log\log n$ increases very slowly therefore

$$O_k(n)>(1-\varepsilon)\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n$$

holds for all but $O\left(xx_1^{-\frac{\varepsilon_1}{3}}\right)$ integers $n\in\left[\frac{x}{2},x\right]$. This assertion holds for $x\geqq X_0$. Choosing now $x=2^\nu X_0$ $(\nu=0,1,\dots)$ and using our result, we obtain (1.2).

The proof of (1.3) is very similar. Since $\bar\varrho(\lambda)\sim 1-\sqrt{2\lambda}$ $(\lambda\sim 0)$, therefore (1.3) is obvious if $\alpha\leqq\frac{\varepsilon^2}{3}$.

Let $\mathcal{P}$ be the set of primes in $(k, x^{1/4r})$,

$$\alpha=\frac{\log k}{x_2},\quad \frac{\varepsilon^2}{3}\leqq\alpha\leqq 1,$$

r be an integer for which $r=Hx_2+O(1)$, $H=\bar\varrho(\alpha)+\varepsilon_3$.

Let $B_{k,r}(x)$ be the number of $n\leqq x$, for which $o_k(n|\mathcal{P})>r$. For these n's $F_k(n)=0$, and by Lemma 4 we get

$$B_{k,r}(x)\leqq c_3 x\left(1+\frac{1}{\Lambda}\right)\left(e^{-r}+\frac{1}{k\log k}\right)+c_4\frac{k^3+k^2\Lambda}{\Lambda^2}. \tag{3.9}$$

From Stirling formula

$$\log \Lambda = \log\left(ke^{-p}\frac{P^r}{r!}\right) = \left(\alpha-1-H\log\frac{H}{e}\right)x_2+O(x_3) = \left(\alpha-\psi(H)\right)x_2+O(x_3).$$

Since $-\psi'(z)=-\log z$ is decreasing,

$$\psi(\bar\varrho)-\psi(H) = \int_{\bar\varrho}^{H} -\log z\, dz \geqq (H-\bar\varrho)\log\frac{1}{H} = \varepsilon_3\log\frac{1}{H},$$

consequently

$$\alpha-\psi(H) = \psi(\bar\varrho)-\psi(H) \geqq \varepsilon_3^2 \quad \text{in} \quad \alpha\in\left[-\frac{\varepsilon^2}{3}, 1\right],$$

if ε_3 is sufficiently small.

Thus $\Lambda \geqq 1$, and

$$B_{k,r}(x) \leqq c_5 x(e^{-r}+k^{-1}). \tag{3.10}$$

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the set of primes in the intervals $[1, k]$, $[x^{1/4r}, x]$, respectively, and

$$P_1 = \sum_{p<k} 1/p = \log\log k+O(1), \quad P_2 = \sum_{x^{1/4r}<p\leqq x} 1/p\log 4r+O(1).$$

Applying Lemma 4 by

$$(L=)\, L_1 = \frac{4\log k}{\log\log k},$$

we get

$$B_k(x, L_1|\mathscr{P}_1) \leqq x/k^3. \tag{3.11}$$

Observing that $\log k=\alpha x_2 \geqq \frac{\varepsilon^2}{3}x_2$, and $P_2=O(x_3)$, by choosing $L=L_1$, we get

$$B_k(x, L_1|\mathscr{P}_2) \leqq c(\varepsilon)\frac{x}{k^3}. \tag{3.12}$$

Since

$$o_k(n) \leqq o_k(n|\mathscr{P})+O_k(n|\mathscr{P}_1)+O_k(n|\mathscr{P}_2),$$

from (3.10), (3.11), (3.12) we have that for large x

$$o_k(n) \leqq r+2L_1 \leqq \left(\bar\varrho(\alpha)+2\varepsilon_3\right)x_2, \tag{3.13}$$

apart from at most

$$c_1(\varepsilon)x\{e^{-(\bar\varrho(\alpha)+\varepsilon_3)x_2}+e^{-\alpha x_2/2}\} \tag{3.14}$$

n in $[1, x]$.

Let $\alpha_t=t\frac{\varepsilon^2}{12}$ $(t=1, \dots, T)$, $T=\left[\frac{12}{\varepsilon^2}\right]+1$, $k_t=[x_1^{\alpha_t}]$. From (3.13) and (3.14) we deduce that

$$o_{k_j}(n) \leqq \left(\bar\varrho(\alpha_j)+2\varepsilon_3\right)x_2 \quad (j=1, \dots, T) \tag{3.15}$$

holds for all but $c_2(\varepsilon)xe^{-\varepsilon_3 x_2}$ n in $[1, x]$, assuming that ε_3 is sufficiently small.

(3.15) easily implies that

$$o_k(n) \leqq \left(\bar{\varrho}(\alpha_j)+\frac{3\varepsilon}{4}\right)x_2 \tag{3.16}$$

for every $k\in[k_1, k_T]$. This is an immediate consequence of the fact that $0\leqq\bar{\varrho}(\alpha_j)-$ $-\bar{\varrho}(\alpha_{j+1})<\frac{\varepsilon}{4}$. Indeed, since $\psi'(2)=\log z$, $-\bar{\varrho}'$ is increasing, we get

$$\bar{\varrho}(\alpha_j)-\bar{\varrho}(\alpha_{j+1})\leqq-\bar{\varrho}'(\alpha_1)\frac{\varepsilon^2}{12}=-\frac{1}{\log\bar{\varrho}(\alpha_1)}\frac{\varepsilon^2}{12}\sim-\frac{1}{\log(1-\sqrt{2\alpha_1})}\frac{\varepsilon^2}{12}<\frac{\varepsilon}{4}.$$

Putting $\log\log n$ instead of x_2 in (3.16), we get that

$$o_k(n)\leqq\left\{\bar{\varrho}\left(\frac{\log k}{\log\log n}\right)+\varepsilon\right\}\log\log n \tag{3.17}$$

holds for all but $c_2(\varepsilon)xe^{-\varepsilon_3 x_2}$ n in $\left[\frac{x}{2}, x\right]$.

Choosing a large X_0 and putting $x=2^\nu X_0$ $(\nu=0, 1, \dots)$ we get (1.3) immediately. Theorem 1 is proved.

4. A counter example. Now we give a non-negative strongly additive $g(n)$ for which $g(p)$ is monotonic, $\sum\frac{g(p)}{p}<\infty$, and (1.5) does not hold.

Let $R_1=1$, $R_{s+1}=\exp(\exp(R_s))$, $J_s=[R_s, R_{s+1})$. We define g for primes p as follows:

$$g(p)=\frac{1}{R_s s^2}\quad(p\in J_s),\ s=1,2,\dots.$$

Sincc

$$\sum_{A<p<B}\frac{1}{p}=\log\frac{\log B}{\log A}+O\left(\frac{1}{\log A}\right),$$

therefore

$$\sum_p\frac{g(p)}{p}=\sum_{s=1}^{\infty}\frac{1}{R_s s^2}\left\{\sum_{p\in J_s}\frac{1}{p}\right\}\ll\sum\frac{1}{s^2}=O(1).$$

Let μ be a large integer, $\mathscr{P}$ be the set of all primes in $(k, R_{\mu+2}]$. Let

$$r=2R_{\mu+1}^2,\quad\log k=(2+\tau)R_{\mu+1}^2\log R_{\mu+1},\quad\frac{1}{4}\leqq\tau\leqq\frac{1}{2}.$$

Let $x\geqq R_{\mu+5}$.

Now we use Lemma 3. Its conditions are fulfilled. By an easy computation we get

$$\sum_{\substack{n\leqq x\\ F_k(n)=0}}1\leqq xe^{-R_{\mu+1}^2} \tag{4.1}$$

for large μ.

Let δ be small, μ be so large that $\delta > e^{-R_{\mu+1}^2}$. Then for all but δx n in $[1, x]$ $F_k(n) \neq 0$. For such an n for at least one j, $1 \leqq j \leqq k$, $n+j$ has at least r prime factors in $[1, R_{\mu+2})$, and so

$$g(n+j) \geqq \frac{r}{R_{\mu+1}(\mu+1)^2}.$$

Consequently

$$f_k(n) \geqq \frac{r}{R_{\mu+1}(\mu+1)^2} = \frac{2R_{\mu+1}}{(\mu+1)^2}. \tag{4.2}$$

Consider now $f_k(0)$. Let t_k be defined as above, i.e. $p_1 \dots p_{t_k} \leqq k \leqq p_1 \dots p_{t_k} p_{t_k+1}$. It is obvious that $f_k(0) = g(t_k)$. From the prime number theorem we get

$$\log k \sim p_{t_k} \sim t_k \log t_k \quad (\mu \to \infty).$$

Let

$$A_s = \prod_{p \in J_s} p \quad (s = 1, \dots, \mu), \quad B = \prod_{R_{\mu+1} \leqq p \leqq p_{t_k}} p.$$

Then

$$g(A_s) = \frac{1}{R_s s^2} \{\pi(R_{s+1}) - \pi(R_s)\},$$

and so

$$\sum_{s=1}^{\mu} g(A_s) \leqq 2 \sum_{s=1}^{\mu} \frac{R_{s+1}}{R_s s^2} \leqq \frac{3R_{\mu+1}}{R_\mu \mu^2}.$$

Furthermore, for an arbitrary but fixed $\varepsilon > 0$

$$g(B) = \frac{1}{R_{\mu+1}(\mu+1)^2} \{\pi(p_{t_k}) - \pi(R_{\mu+1})\} \leqq \frac{t_k}{R_{\mu+1}(\mu+1)^2} \leqq$$

$$\leqq (1+\varepsilon) \frac{\log k}{(\log\log k) R_{\mu+1}(\mu+1)^2} \leqq (1+\varepsilon)\left(1+\frac{\tau}{2}\right) \frac{R_{\mu+1}}{(\mu+1)^2},$$

if μ is sufficiently large. Consequently for large μ

$$f_k(0) < 1{,}6 \frac{R_{\mu+1}}{(\mu+1)^2}, \quad \text{and} \quad f_k(m) > 2 \frac{R_{\mu+1}}{(\mu+1)^2}$$

for all but δx of n's in $[1, x]$.

5. Proof of Theorem 2. Suppose that the conditions (1.7), (1.8) are fulfilled. If $A(y)$ is bounded then the assertion is almost obvious. Indeed, if $A(\infty) = B$, then $\sup g(n) = B$, i.e. $f_k(n) \leqq B$. Furthermore $f_k(0) \to B$, and so $f_k(n) - f_k(0) < \varepsilon f_k(0)$ for every n, if k is large enough.

Suppose now that $A(y) \to \infty$ $(y \to \infty)$. Observe that the prime number theorem easily implies

$$f_k(0) = (1+o(1)) A(\log k) \quad k(\to \infty). \tag{5.1}$$

Furthermore from $t(y) \to 0$ $(y \to \infty)$ we obtain

$$f_{2k}(0) = f_k(0) + o(1) = (1+o(1)) f_k(0). \tag{5.2}$$

Hence

$$f_k(0) \leqq f_k(n) \leqq f_{k+x}(0) \leqq f_{2k}(0) \leqq (1+\varepsilon) f_k(0), \tag{5.3}$$

if $k > x$, $n \leqq x$, k is large.

Now we assume that $k \leqq x$. Let δ be small,

$$H = \exp(\exp((\log k)^\delta)),$$

and

$$g_1(p) = \begin{cases} g(p), & \text{if} \quad p \leqq H, \\ 0, & \text{if} \quad p > H; \end{cases}$$

$$g_2(p) = \begin{cases} 0, & \text{if} \quad p \leqq H, \\ g(p), & \text{if} \quad p > H, \end{cases}$$

and $g_1(n)$, $g_2(n)$ are the corresponding additive functions. Let

$$f_k^{(i)}(n) = \max_{j=1, \dots, k} g_i(n+j) \quad (i = 1, 2).$$

It is obvious that

$$f_k(n) \leqq f_k^{(1)}(n) + f_k^{(2)}(n).$$

Let $\theta = 1 + 2\delta$,

$$r = \left[\theta \frac{\log k}{\log \log k}\right].$$

Let $C_r(x)$ be the number of those $n \leqq x$ that have at least r prime divisors in $[1, H]$. It is obvious that

$$C_r(x) \leqq \sum_{t_r} \left[\frac{x}{t_r}\right] \leqq \frac{xP^r}{r!}, \quad P = \sum_{p \leqq H} \frac{1}{p}.$$

We have

$$kC_r(x) \leqq x \exp\left(\log k - r \log \frac{r}{Pe} + O(\log r)\right),$$

and by

$$P = (\log k)^\delta + O(1)$$

we get

$$\log k - r \log \frac{r}{Pe} + O(\log r) \leqq -\frac{\delta}{4} \log k,$$

i.e.

$$kC_r(x) \leqq \frac{x}{k^{\delta/4}}. \tag{5.4}$$

If the integers $n+j$ $(j = 1, \dots, r)$ have no r distinct prime factors from $[1, H]$, then

$$f_k^{(1)}(n) \leqq g(p_1 \cdots p_{r-1}) \leqq (1+3\delta) A(\log k).$$

Thus we proved that

$$f_k^{(1)}(n) < (1+3\delta) A(\log k)$$

for all but $x/k^{\delta/4}$ integers $n \in [1, x]$.

Let now η be a small positive constant, $\Delta=\eta A(\log k)$. We put $z=e^u$ $(u\geqq 0)$,

$$D(x, z) = \sum_{n\leqq x} z^{g_2(n)}.$$

The function $z^{g_2(n)}$ is multiplicative, and its Moebius transform $l(n)$ is defined for prime powers as

$$l(p) = \begin{cases} e^{ug(p)}-1, & p > H, \\ 0, & p < H, \end{cases}$$

$l(p^\alpha)=0\ (\alpha\geqq 2)$.

Consequently

$$D(x, z) = \sum_{d\leqq x} l(d)\left[\frac{x}{d}\right] \leqq x \prod_{H<p\leqq x}\left(1+\frac{e^{ug(p)}-1}{p}\right).$$

Let $u=\dfrac{1}{2t(H)}$. Then from $e^{ug(p)}-1<2ug(p)$ it follows that

$$D(x, z) \leqq x\exp\left(2u\sum_{H<p<x}\frac{t(p)}{p}\right).$$

Let $B(x, \eta, k)$ denote the number of those $n\leqq x$, for which $f_2(n)\geqq\Delta$. We obtain

$$B(x, \eta, k) \leqq k\sum_{n\leqq x} z^{g_2(n)-\Delta u} \leqq x\exp\left(-\Delta u+2u\sum_{H<p<x}\frac{t(p)}{p}+\log k\right).$$

From (1.9) we have

$$-\Delta u+2u\sum_{H<p<x}\frac{t(p)}{p}+\log k < -3\log k$$

for large k, i.e.

$$B(x, \eta, k) \leqq \frac{x}{k^3}.$$

Consequently

$$f_k(n) < (1+3\delta+\eta)A(\log k)$$

for all but $\left(\dfrac{1}{k^{\delta/4}}+\dfrac{1}{k^3}\right)x$ integers n in $[1, x]$, for every large k. Let $3\delta+\eta<\dfrac{\varepsilon}{4}$. From (5.1) we get

$$f_k(n) < \left(1+\frac{\varepsilon}{2}\right)f_k(0), \tag{5.6}$$

if $k\geqq c(\varepsilon)$.

We choose $(k=)k_\nu=2^\nu k_0$ $(\nu=0, 1, 2, \ldots)$. Then

$$f_{k_\nu}(n) < \left(1+\frac{\varepsilon}{2}\right)f_k(0) \quad (\nu=0, 1, 2, \ldots), \tag{5.7}$$

allowing at most

$$2x\sum_{\nu=1}^{\infty} k_\nu^{-\delta/4} \leqq \frac{cx}{k_0^{\delta/4}}$$

integers n in $[1, x]$. Suppose that (5.7) holds for an n. If $k \geqq k_0$ $k \in [k_\nu, k_{\nu+1})$, then from

$$f_k(n) \leqq f_{k_{\nu+1}}(n) \leqq \left(1+\frac{\varepsilon}{2}\right) f_{k_{\nu+1}}(0) < \left(1+\frac{\varepsilon}{2}\right)\left(1+\frac{\varepsilon}{4}\right) f_k(0),$$

the inequality

$$f_k(n) < (1+\varepsilon) f_k(0)$$

follows for every $k \geqq k_0$, which completes the proof of Theorem 2.

References

[1] I. Kátai, Local growth of the number of prime divisors of consecutive integers, *Publ. Math. Debrecen*, **18** (1971), 171—175.
[2] I. Kátai, Maximum of number-theoretical functions in short intervals, *Annales Univ. Sci. Budapest, Sectio. Math.*, **18** (1975), 69—74.
[3] A. Iványi and I. Kátai, On monotonic additive functions, *Acta Math. Acad. Sci. Hungar.*, **24** (1973), 203—208.
[4] R. A. Rankin, The difference between consecutive prime numbers, *J. London Math. Soc.*, **13** (1938), 242—247.
[5] H. Halberstam and H. E. Richert, *Sieve methods*, Academic Press (London, 1974).
[6] H.-E. Richert, Selberg's sieve, *Problems in analytical number theory*, Mir (Moscow, 1975).

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