Journal of the Indian Math. Soc.
41 (1977) 281–291

AN ASYMPTOTIC FORMULA IN ADDITIVE NUMBER THEORY—II

By P. ERDÖS, G. JOGESH BABU AND K. RAMACHANDAR

[Received February 13, 1976; revised November 30, 1976]

This paper is in some sense a sequel to our earlier paper I (Acta. Arith; 28 (1976), 405–412) with the same title although the present paper is self contained.

Let $\{b_j\}$ be an increasing sequence of integers with $3 \leqslant b_1 < b_2 < b_3 \dots$ and $\sum_j \frac{1}{b_j} < \infty$. Our principal object is to prove, under an assumption on the size of $B(x) = \sum_{b_j \leqslant x} 1$, that for any fixed position integer n, the number of solutions of the equation $n = p + t$ where t is a positive integer not divisible by any b_j and p is a prime exceeds $\alpha\, n/\log n + o\,(n/\log n)$, where α is a positive constant, and in particular $\geqslant 1$ for all sufficiently large n. (The assumption on $B(x)$ is $B(x) = o\left(\frac{x}{\log x \log \log x}\right)$. It will be clear from our proof that this can be weakened to $B(x) = o\left(\frac{x}{\log x}\right)$ if a certain unproved hypothesis on the distribution of primes in arithmatic progressions is true. We prefer to state this hypothesis at the end of our proof).

Before starting the proof proper we make some reductions. Consider those b_j with $\frac{b_j}{\varphi(b_j)} \geqslant 100$. For these b_j we have $\frac{\sigma(b_j)}{b_j} \geqslant 2$ (φ is the Euler's totient function and σ is the sum of the divisors) and so such b_j are abundant numbers (m is said to be abundant if $\frac{\sigma(m)}{m} \geqslant 2$). It is easy to see that every multiple of an abundant number is also abundant. Defining an adundant number N to be primitive if N is the only abundant number which divides N we have the following:

THEOREM (due to P. Erdös, On the density of abundant numbers, Jour. London Math. Soc. IX (1934), pp. 278–282, see theorem on page 281). *The number of primitive abundant numbers not exceeding x is* $o\left(\frac{x}{(\log x)^2}\right)$.

From $\{b_j\}$ construct a new sequence by retaining as they are numbers b_j with $\frac{b_j}{\varphi(b_j)} \leqslant 100$ and replacing every other number by its maximum primitive abundant divisor. From the resulting set form a sequence in the increasing order by taking only the distinct ones. Suppose this sequence is $\{b'_j\}$ where $3 \leqslant b'_1 < b'_2 < b'_3 \ldots$. (This sequence consists of un-replaced and replaced numbers of $\{b_j\}$). Note that $\sum \frac{1}{\varphi(b'_j)}$ is convergent. Because $\sum\limits_{X,\,2X} \frac{1}{\varphi(b'_j)}$ (this sum is over all b'_j satisfying $X \leqslant b'_j \leqslant 2X$ and we adopt a similar notation elsewhere) $= \sum\limits_1 + \sum\limits_2$ where $\sum\limits_1$ is part of the original $\sum\limits_{X,\,2X}{}^{*} \frac{1}{\varphi(b_j)}$ without replacements and $\sum\limits_2$ the rest. In $\sum\limits_1$ $\frac{b_j}{\varphi(b_j)} \leqslant 100$ and so $\sum\limits_1 = O\left(\sum\limits_{X,\,2X} \frac{1}{b_j}\right)$ and $\sum\limits_2 = O\left(\sum\limits_{X,\,2X}{}' \frac{\log\log b'_j}{b'_j}\right)$ where $\sum\limits_{X,\,2X}{}'$ denotes the restriction to the altered numbers and so $\sum\limits_2$ $= o\left(\frac{\log\log X}{(\log X)^2}\right)$ and this gives us the convergence of the required series.

Not let $\{d_i\}$ be the sequence $1 = d_1 < d_2 < d_3 \ldots$ of integers not divisible by any b_j and $\{d'_1\}$ the sequence which corresponds to $\{b'_j\}$ is a similar fashion. The sequence $\{d_1\}$ includes $\{d'_1\}$ and so the number of solutions of $n = p + d_j$ is at least the number of solutions of $n = p + d'_j$. We prove for the latter number a lower bound $\gg \frac{n}{\log n}$ valid for all large enough n. It follows that the number of solutions of $n = p + d_j$ is also $\gg \frac{n}{\log n}$ for all large enough n. This is in fact the principal result we are looking for. (We however assume only at one place of our proof that $B(x) = o\left(\frac{x}{\log x \log\log x}\right)$ and at this point of our proof even the weaker assumption $B(x) = o\left(\frac{x}{\log x}\right)$ would suffice if we assume the truth of an unproved conjecture concerning the distribution of primes in arithmetic progressions). Note that $\sum_{b'_j \leqslant x} 1 = \sum_3 + \sum_4$ where $\sum_3$ counts the unreplaced b_j and $\sum_4$ counts the replaced ones and so $\sum_3 = O(B(x))$ and $\sum_4 = o\left(\frac{x}{(\log x)^2}\right)$. We may also note $\sum_{X, 2X} \frac{1}{\varphi(b'_j)} = \sum_5 + \sum_6$ where $\sum_5 = O\left(\sum_{X, 2X} \frac{1}{b_j}\right)$ and $\sum_6 = o\left(\frac{\log\log X}{(\log X)^2}\right)$ are obvious portions of the sum. From now on we write a_i to mean b'_i and write $A(x) = \sum_{a_i \leqslant x} 1$. Throughout the paper we assume $B(x) = o\left(\frac{x}{\log x}\right)$ which certainly gives $A(x) = o\left(\frac{x}{\log x}\right)$. We now start the proof proper. We find it convenient to split it into several parts.

PART I. *Estimation of* $\sum_{X_1 \leqslant a_i \leqslant n} \sum_{\substack{p \equiv n \pmod{a_i} \\ 1 < p < n}}$ *for a suitable* X_1.

Denote the inner sum by $\pi(n, a_i)$ and consider $\sum_{n/2^{k+1} \leqslant a_i \leqslant n/2^k} \pi(n, a_i)$ for a given $k = 0, 1, 2, \ldots$. We wish to estimate this uniformly in all parameters including k. For any given k the sum is $\sum_{a_i} O(2^k)\ (= O(A(n))$ for bounded k). Because trivially $\pi(n, a_i) = O(2^k)$. Thus fixing up any arbitrarily large constant k_0, we have,

$$\sum_{0 \leqslant k \leqslant k_0} \sum_{n/2^{k+1} \leqslant a_i \leqslant n/2^k} \pi(n, a_i) = o\left(\frac{n}{\log n}\right).$$

We now introduce the points $2^{k/n}$ $(k = 0, 1, 2, \ldots)$ and split up the range $X_1 \leqslant a_i \leqslant n$ accordingly with proper modification at the end points. We have now to estimate $S_1 = \sum_{X_1 \leqslant a_i \leqslant n/2^{k_0}} \pi(n, a_i)$.

The contribution to this sum from those a_i (We now fix up till the end of the proof small positive constants ε, δ, δ_1 which are arbitrary but independent of each other) satisfying,

$$(n, a_i) \leqslant \frac{n}{a_i (\log n/a_i) (\log \log n/a_i)^{1+\varepsilon}}$$

is

$$o\left(\sum_{k \geqslant k_0,\, n/2^{k+1} \geqslant X_1} \left(\frac{2^k}{k(\log k)^{1+\varepsilon}} \cdot \frac{n}{2^k \log(n/2^k)} \right) \right) = o\left(\frac{n}{\log n} \right)$$

$$n \geqslant X_1 \geqslant n^{1-\delta}.$$

Consider the remaining portion S_2 of the sum S_1. We are led to estimate

$$\sum^{*}_{n/2^{k+1} \leqslant a_i \leqslant n/2^k} \pi(n, a_i)$$

where $*$ denotes the restriction to those a_i which satisfy.

$$\pi(n, a_i) \geqslant \frac{2^k}{k(\log k)^{1+\varepsilon}}.$$

First consider the contribution to S_2 from those k for which the number of a_i does not exceed $\frac{\delta_1 n}{2^k k \log n}$. We observe that the contribution from such an integer k is by Brun–Titchmarsh Theorem $O\left(\sum_{a_i} \frac{n}{\varphi(a_i) \log(n/a_i)} \right)$ where the sum over a_i is over an appropriate range depending on k. We split this last sum into two parts according as a_i is unchanged or changed and we see that it is

$$O\left(\frac{n}{a_i \log(n/a_i)} \right) + o\left(\frac{n}{2^k (\log(n/2^k))^2} \cdot \frac{2^k \log\log(n/2^k)}{\log(2^k)} \right)$$

$$= O\left(\frac{\delta_1 n}{2^k\, k \log n} \cdot \frac{2^k}{\log(2^k)} \right) + o\left(\frac{n \log\log(n/2^k)}{k(\log(n/2^k))^2} \right)$$

$$= O\left(\frac{\delta_1 n}{k^2 \log n} \right) + (\ldots)$$

It is easy to see that the last expressions when summed over from $k = k_0$ to $[2\delta \log n]$ is $O\left(\frac{\delta_1 n}{\log n} \right) + o\left(\frac{n}{\log n} \right)$.

So far we imposed on X_1 the only condition $n \geqslant X_1 \geqslant n^{1-\delta}$. We now show that if X_1 is properly chosen there do not exist any other values of k which make a further contribution to S_2.

So we have now to consider only those k for which the number of

a_i's is $\geqslant \frac{\delta_1 n}{2^k k \log n}$ and each a_j satisfies $\pi(n, a_i) \geqslant \frac{2^k}{k(\log k)^{1+\varepsilon}}$. For such a fixed k let s be the number of a_i. Let us enumerate these a_i (with a change of notation to avoid too many symbols) as

$$\frac{n}{2^{k+1}} \leqslant a_1 < a_2 < \ldots < a_s \leqslant \frac{n}{2^k}$$

where

$$s \geqslant \frac{\delta_1 n}{2^k k \log n} \text{ and } Z_i = \pi(n, a_i) \geqslant \frac{2^k}{k(\log k)^{1+\varepsilon}}.$$

Write

$$n - p_j^{(i)} = t_j^{(i)} a_i \text{ where } 1 \leqslant t_j^{(i)} \leqslant 2^{k+1}.$$

For any fixed i the number of pairs $(p_{j_1}^{(i}, p_{j_2}^{(i)})$, $(j_1 \neq j_2)$ is $\geqslant \binom{Z_i}{2} \geqslant \frac{4^k}{4k^2(\log k)^{2+2\varepsilon}}$ and there are $s \geqslant \frac{\delta_1 n}{2^k k \log n}$ values of i. Hence the total number of pairs is

$$\geqslant \sum_i \binom{Z}{2^i} \geqslant \frac{\delta_1 n 2^k}{4k^3(\log k)^{2+2\varepsilon} \log n}.$$

Let t_1, t_2 be integers satisfying $t_1 \neq t_2$, $1 \leqslant t_1 \leqslant 2^{k+1}$ and $1 \leqslant t_2 \leqslant 2^{k+1}$. It follows that if $N(t_1, t_2)$ denotes the total number of triplets (i, j_1, j_2) with $t_{j_1}^{(i)} = t_1$, $t_{j_2}^{(i)} = t_2$ then

$$\sum_{(t_1, t_2)\ t_1 \neq t_2,} \pi(t_1, t_2) \geqslant \frac{\delta_1 n 2^k}{4k^3(\log k)^{2+2\varepsilon} \log n}.$$

The total number of pairs (t_1, t_2) does not exceed $2^{2(k+1)}$ and hence there exists a pair (t_1, t_2) (of course $t_1 \neq t_2$ and $1 \leqslant t_1 \leqslant 2^{k+1}$, $1 \leqslant t_2 \leqslant 2^{k+1}$) such that the simultaneous equations $n - p_1 = t_1 a$, $n - p_2 = t_2 a$ (where a is a positive integer) have

$$\geqslant \frac{\delta_1 n}{2^k k^3 \log n (\log k)^{2+2\varepsilon}} (= Q \text{ say})$$

solutions in triplets (p_1, p_2, a). That is, there are $\geqslant Q$ values of a $(1 \leqslant a \leqslant n/2^k,\ 2^k \leqslant n^\delta)$ for which $n - a t_1$ and $n - a t_2$ are both primes. By the double sieve, the number of such integers a is $O\left(\frac{n(\log\log n)^2}{2^k(\log n)^2}\right)$ (see page 45, Satz 4.2 of Prachar's book). This gives

$$\frac{\delta_1 \log n}{k^3(\log k)^{2+2\varepsilon}} = O((\log\log n)^2.)$$

This gives a contradiction for large n if $k \leqslant \left(\frac{\log n}{(\log\log n)^{4+3\varepsilon}}\right)^{1/3}$.

So we can choose X_1 to be

$$X_1 = n\, 2^{-\left[(\log n)^{1/3} (\log\log n)^{-\frac{4+3\epsilon}{3}}\right]}$$

This completes the proof that

$$\sum_{X_1 \leqslant a_i \leqslant n} \pi(n, a_i) = O\left(\frac{\delta_1 n}{\log n}\right) + o\left(\frac{n}{\log n}\right)$$

where X_1 is chosen as stated just now (actually since the left side is independent of δ_1 the first term on the right can be dropped).

PART II. *Estimation of*

$$\sum_{i \geqslant L,\ a_i \leqslant n^{1-\delta}} \pi(n, a_i)$$

where $\delta > o$ *is fixed and* L *is a large constant.*

Applying Brun-Titchmarsh theorem the estimate for the required sum

$$= O\left(\sum_{i \geqslant L} \frac{n}{\varphi(a_i) \log n}\right) = O\left(\frac{n}{\log n}\, \eta(L)\right)$$

where $\eta(L)$ tends to zero as L tends to infinity because of the convergence of $\Sigma\, (\varphi(a_i))^{-1}$.

PART III. *Estimation of*

$$\sum_{n^{1-\delta} \leqslant a_i \leqslant n \operatorname{Exp}(-(\log n)^{\frac{1}{3}} (\log\log n)^{-2})} \pi(n, a_i).$$

We split up the range into minimum number intervals of the type $X \leqslant a_i \leqslant 2X$ with modification at the end points and write it in the form $\sum\limits_{X} \sum_X$. Each $\sum_X$ can be written $\overset{(1)}{\sum_X} + \overset{(2)}{\sum_X}$ where (1) is over those a_i with

$$\pi(n, a_i) \leqslant \frac{10^8\, n}{\varphi(a_i) \log n}$$

and (2) is over the remaining a_i. By the convergence of $\Sigma\, (\varphi(a_i))^{-1}$ we have easily

$$\sum_X \overset{(1)}{\sum_X} = o\left(\frac{n}{\log n}\right).$$

$\sum\limits_X^{(2)}$ is by Brun-Titchmarsh theorem

$$O\left(\sum_X^{(2)} \frac{n}{\varphi(a_i)\log\left(\frac{n}{a_i}\right)}\right) = o\left(\sum_X^{(2,1)} + \sum_X^{(2,2)}\right)$$

where (2, 1) is over unchanged a_i and (2, 2) is over the replaced ones. Trivially

$$\sum_X \sum_X^{(2,2)} = \sum_X O\left(\frac{n \log\log X}{(\log X)^2 \log\left(\frac{n}{X}\right)}\right) = o\left(\frac{n}{\log n}\right).$$

Let $A^{(1)}(X)$ be the number of unchanged a_i lying between X and $2X$ for which

$$\pi(n, a_i) \geqslant \frac{10^8 n}{\varphi(a_i)\log n}.$$

Then

$$\sum_X \sum_X^{(2,1)} = O\left(\sum_X \frac{A^{(1)}(X)}{X} \frac{n}{\log\left(\frac{n}{X}\right)}\right).$$

By using a bound of the type $A^{(1)}(X) = O(A(X))$ we can easily prove that the last quantity is $o\left(\frac{n}{\log n}\right)$ if we assume $A(X) = o\left(\frac{X}{\log X \log\log X}\right)$. On the other hand we can also majorise $A^{(1)}(X)$ by $A^{(2)}(X)$ the number of all integers q satisfying $X \leqslant q \leqslant 2X$ and $\pi(n, q) \geqslant \dfrac{10^8 n}{\varphi(q)\log n}$ and make the

Hypothesis. *Uniformly in* $n^{1-\delta} \leqslant X < 2X \leqslant n\, Exp\left(-\frac{\log n)^{1/3}}{(\log\log n)^2}\right)$ *there holds* $A^{(2)}(X) = O(X(\log X)^{-5/3-\delta_2})$ *for some constant* $\delta_2 > 0$.

We see on replacing $\log\frac{n}{X}$ by $(\log X)^{1/3}(\log\log X)^{-2}$, that

$$\sum_X \sum_X^{(2,1)} = o\left(\frac{n}{\log n}\right).$$

Part IV. *Lower bound for the number of solutions of* $n = p + d_j$.

Write A_L for the finite sequence $(a_1, a_2, \ldots, a_L)$ and for any positive integer n write (In this section t will stand for a positive integer)

$$f(n, A_L) = \sum_{n = p+t,\, t \not\equiv 0 \pmod{a_j} \text{ for } j = 1 \text{ to } L} 1.$$

We use a similar notation for any other finite or infinite sequence of positive integers in place of A_L. We choose L to be a large but fixed integer and another fixed positive integer $h < L$ then certain odd primes $q_i\,(1 \leqslant i \leqslant h)$, in the following way. Since a_j is never less than 3, it is divisible either by 4 or by an odd prime q_j (in the latter case we fix q_j to be the least odd prime which divides a_j). If in this process 4 occurs we designate it by q_0 and if it does not occur we just ignore the symbol q_0. Of course $q_j\,(j = 1$ to $h)$ need not be distinct. Let A^* denote the finite sequence $(q_0, q_1, q_2, \ldots, q_h, a_{h+1}, a_{h+2}, \ldots, a_L)$. Before proceeding further it may be helpful to remark that $f(n, A_L) \geqslant f(n, A^*)$. For simplicity we write A^{**} for the sequence obtained from A^* by retaining only the distinct $q_j\,(1 \leqslant j \leqslant h)$. Next in A^{**} retain only those $q_j\,(1 \leqslant j \leqslant h)$ which do not divide n and afterwards only those $a_j\,(j > h)$ with $(a_j, \prod_{1 \leqslant i \leqslant h} q_i) = 1$. Call the resulting set

$$\bar{A} = (q_0, q_1, \ldots, q_J, a'_{J+1}, a'_{J+2}, \ldots, a'_T)$$

where the notation is sufficiently self-explanatory. Let S^* and S^{**} be two finite sets of distinct integers and 1 be an element of S^{**}. We observe that the set

$$S^* \cap (S^{**} - 1)$$

has at least as many elements as -1 plus the number of elements in $S^* \cap S^{**}$. Using this remark repeatedly one can verify that

$$f(n, A_L) \geqslant f(n, A^+) \geqslant f(n, q_J, q_1, q_2, \ldots, A, a'_{J+1}, a'_{J+2}, \ldots, a'_T) - J.$$

We now make the convention that $q_j\,(1 \leqslant j \leqslant J)$ are in the increasing order. We next replace all $a'_j\,(j > J)$ which are even but not multiples of 4 by $\frac{1}{2}a'_j$ and designate the set resulting from $a'_j\,(J < j \leqslant T)$ in the increasing order by $a''_j\,(J < j \leqslant T)$. Our last lower bound for $f(n, A_L)$ is

$$\geqslant f(n, 4, q_1, q_1, \ldots, q_J, a''_{J+1}, a''_{J+2}, \ldots, a''_T) - J$$
$$\geqslant f(n, 4, q_1, q_2, \ldots, q_J) - J$$
$$- \sum_{\nu = J+1}^{T} \quad \sum_{n = p+t,\, t \not\equiv 0 \bmod q_j \text{ for all } j \text{ in } (o \leqslant j \leqslant J),\, t \equiv 0 \pmod{a''_\nu}} 1.$$

Here (and from now on) we put $q_0 = 4$. Note that the present q_0 always denotes 4 whether the old q_0 already introduced may or may not exist. This will not cause any confusion since the purpose of introducing the old q_0 is over and we do not need it any more. By using the prime number theorem for arithmetic progressions and a simple argument of Eratosthanes it is not hard to verify the following steps (the notations are obvious and we do not explain them)

$$f(n, q_0, q_1, \ldots, q_J) = \pi(n) - \sum_i \pi(n, q_i, n) + \sum_{i \neq j} \sum_{i > j} \pi\,(n, q_i q_j, n) - + \ldots$$
$$= \frac{n}{\log n} \prod_{0 \leqslant i \leqslant J} \left(1 - \frac{1}{\varphi(q_i)}\right) + O_h\left(\frac{n}{(\log n)^2}\right)$$

In the sum over ν the ν-th term is

$$= \pi(n, a''_\nu, n) - \sum_i \pi(n, [a''_\nu, q_i], n)$$
$$+ \sum_{1 \neq j,\, i > j} \pi(n, [a''_\nu, q_i q_j], n) - + \ldots$$
$$= \frac{n}{\varphi(a''_\nu) \log n} \prod_{o \leqslant i \leqslant J} \left(1 - \frac{1}{\varphi(q_i)}\right) + O_{h,\, a''_\nu}\left(\frac{n}{(\log n)^2}\right)$$

Thus $f(n, A_L)$ exceeds

$$\frac{n}{\log n} \prod_{o \leqslant i \leqslant J} \left(1 - \frac{1}{\varphi(q_i)}\right)\left(1 + O\left(\sum_{\nu \geqslant J+1} \frac{1}{\varphi(a''_\nu)}\right)\right) + O_L\left(\frac{n}{(\log n)^2}\right).$$

From our definition of a''_ν and the convergence of $\Sigma \dfrac{1}{\varphi(a_\nu)}$ it follows that the last expression exceeds

$$\frac{Cn}{\log n} + O_L\left(\frac{n}{(\log n)^2}\right)$$

where $C\,(> o)$ is independent of L and n but depends only on h. Now if we fix first a large h and then a larger L, we have

$$f(n, A) \geqslant f(n, A_L) - \sum_{\nu \geqslant L+1} \sum_{\substack{n = p+t \\ t \equiv o \pmod{a_\nu}}} 1$$

$$> \frac{C_1 n}{\log n} \ (C_1 > 0 \text{ independent of } n)$$

provided $n \geqslant n_0$, by the results of parts I, II and III.

PART V. *Statement of the main theorem.* Collecting together we state

THEOREM. *Let* $\{b_j\}\ j = 1, 2, \ldots$ *be a finite or an infinite sequence of of integers satisfying* $3 \leqslant b_1 < b_2 < b_3 \ldots$ *and* $\sum \frac{1}{b_j} < \infty$. *Let* $1 = d_1 < d_2 < d_3 \ldots$ *be the sequence of all integers* d_i $(i = 1, 2, 3, \ldots)$ *which are not divisible by any* b_j. *Let* $B(x) = \sum_{b_j \leqslant x} 1$ *and* $B(x) = o\left(\frac{x}{\log x \log \log x}\right)$. *Then the number solutions for any fixed* $n \geqslant n_0$ *(a large constant depending on the constants implied by the sequence and the nature of* $o(\ldots)$*) of the equation*

$$n = p + d_j \ (p - \text{prime})$$

is $\gg \frac{n}{\log n}$ *and in particular* $\geqslant 1$.

REMARK. The conclusion of the theorem is valid even with the milder assumption $B(x) = o\left(\frac{x}{\log x}\right)$ if the following hypothesis regarding the distribution of primes in arithmetic progressions is true.

HYPOTHESIS. *Let* $\delta > o$ *be any small constant and*

$$n^{1-\delta} \leqslant X < 2X \leqslant n \ Exp\,(-(\log n)^{1/3} (\log \log n)^{-2}).$$

Then the number of integers q *satisfying* $X \leqslant q \leqslant 2X$ *and*

$$\pi(n, q, n) \geqslant \frac{10^8\, n}{\varphi(q) \log n} \text{ is } O_\delta\left(\frac{X}{(\log X)^\lambda}\right)$$

where $\lambda > \frac{5}{3}$ *is a constant.*

The following hypothesis is also sufficient and is perhaps simpler to prove than the one stated above.

HYPOTHESIS. *Let* $n^{1-\delta} \leqslant X < 2X \leqslant n \, Exp\,(-(\log n)^{1/3})$. *Then the number of integers q satisfying* $X \leqslant q \leqslant 2X$ *for which*

$$\pi(n, q, n) \geqslant n\,(\log \log n)^2\,(q \log n)^{-1} \text{ is } o\,(X\,(\log X \log \log X)^{-1}).$$

Mathematical Institute of the
Hungarian Academy of Sciences
Realtanoda U–13–15
BUDAPEST, HUNGARY

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005
INDIA