Reprinted from JOURNAL OF COMBINATORIAL THEORY
All Rights Reserved by Academic Press, New York and London

Vol. 14, No. 1, February 1973
Printed in Belgium

Ramsey Numbers for Cycles in Graphs

J. A. BONDY AND P. ERDÖS

University of Waterloo, Waterloo, Ontario, Canada

Received January 28, 1972

Given two graphs G_1, G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer m such that, for any partition (E_1, E_2) of the edges of K_m, either G_1 is a subgraph of the graph induced by E_1, or G_2 is a subgraph of the graph induced by E_2. We show that

$$\begin{aligned}
R(C_n, C_n) &= 2n-1 \quad \text{if} \quad n \text{ is odd},\\
R(C_n, C_{2r-1}) &= 2n-1 \quad \text{if} \quad n > r(2r-1),\\
R(C_n, C_{2r}) &= n+r-1 \quad \text{if} \quad n > 4r^2-r+2,\\
R(C_n, K_r) &\leqslant nr^2 \quad \text{for all} \quad r, n,\\
R(C_n, K_r) &= (r-1)(n-1)+1 \quad \text{if} \quad n \geqslant r^2-2,\\
R(C_n, K_r^{t+1}) &= t(n-1)+r \quad \text{for large } n.
\end{aligned}$$

1. INTRODUCTION

We are here concerned with undirected graphs that are finite and have no loops or multiple edges. Let G be such a graph; we write $V(G)$ for the vertex set of G, and $E(G)$ for the edge set of G; $|V(G)|$ is the *order* of G, $|E(G)|$ the *size* of G. If $E' \subseteq E(G)$, E' will also denote the partial subgraph of G with edge set E'. C_n denotes the cycle of length n, K_n the complete graph of order n, and $K(r_1, ..., r_t)$ the complete t-partite graph with parts of cardinalities $r_1, ..., r_t$; when each $r_i = r$ this will be written $K_r{}^t$.

Let k be finite and let

$$m \to (G_1, ..., G_k)$$

signify the truth of the statement: for any partition $(E_1, ..., E_k)$ of $E(K_m)$ there is an i, $1 \leqslant i \leqslant k$, such that G_i is a subgraph of E_i. It follows from Ramsey's theorem that, for any collection of graphs $G_1, ..., G_k$, there is a finite m such that $m \to (G_1, ..., G_k)$. We denote the least such m by $R(G_1, ..., G_k)$.

The Ramsey function R has been studied in detail for complete graphs G_i, although exact values are generally unknown. Chvátal and Harary

[3, 4] determined $R(G_1, G_2)$ for all G_1, G_2 of order at most four, and Chartrand and Schuster [2] have shown that

$$R(C_n, C_3) = \begin{cases} 6, & n = 3, \\ 2n - 1, & n > 3, \end{cases}$$

$$R(C_n, C_4) = \begin{cases} 6, & n = 4, \\ 7, & n = 5, \\ n + 1, & n > 5, \end{cases}$$

$$R(C_n, C_5) = 2n - 1, n > 2,$$

$$R(C_6, C_6) = 8.$$

In this paper we investigate $R(C_n, C_r)$ for arbitrary $r \leqslant n$. It was conjectured by W. G. Brown that, for $n > n_0(r)$,

$$2n - 1 \to (C_n, C_r).$$

We prove this (with $n_0(r) = \frac{1}{2}(r^2 + r)$); it follows easily that, for odd r and $n > \frac{1}{2}(r^2 + r)$,

$$R(C_n, C_r) = 2n - 1.$$

It seems likely that, for $n > 3$ and all $r \leqslant n$, $2n - 1 \to (C_n, C_r)$, but we can only prove at present that

$$2n - 1 \to (C_n, C_n), \qquad n > 3.$$

We also show that, for $n > 4r^2 - r + 2$,

$$R(C_n, C_{2r}) = n + r - 1.$$

More generally we prove that, for $n > n_1(r, t)$,

$$R(C_n, K_r^{t+1}) = t(n - 1) + r.$$

This implies that, for $n > n_2(r)$ $(= n_1(1, r - 1))$,

$$R(C_n, K_r) = (r - 1)(n - 1) + 1.$$

In fact we prove directly that the above holds for $n \geqslant r^2 - 2$. Finally we show that, for arbitrary r and n,

$$nr^2 \to (C_n, K_r).$$

2. Preliminary Lemmas

Let $G(r_1, ..., r_t)$ denote the complete graph of order $\sum_{i=1}^{t} r_i$ with edge partition (E_1, E_2) such that $E_2 \cong K(r_1, ..., r_t)$.

LEMMA 1. $R(C_n, C_{2r-1}) > 2n - 2$.

Proof. $G(n-1, n-1)$ contains no C_n in E_1 and no C_{2r-1} in E_2.

LEMMA 2. $R(C_n, K_r^{t+1}) > t(n-1) + r - 1$.

Proof. $G(n_1, ..., n_t, s_1, ..., s_{r-1})$, where $n_i = n - 1$, $1 \leqslant i \leqslant t$, and $s_i = 1$, $1 \leqslant i \leqslant r - 1$, contains no C_n in E_1 and no K_r^{t+1} in E_2.

LEMMA 3 (Erdös and Gallai [5]). *If G is a graph of order n and size at least $\frac{1}{2}((c-1)(n-1)+1)$, then G contains a cycle of length at least c.*

LEMMA 4 (Bondy [1]). *If G is a graph of order n and size at least $\frac{1}{4}(n^2+1)$, then G contains cycles of all lengths l, $3 \leqslant l \leqslant \frac{1}{2}(n+3)$.*

LEMMA 5 (Erdös and Stone [6]). *If G is a graph of order n and size at least $\frac{1}{2}n^2(1 - 1/(t-1) + \epsilon)$, where $n > n(t, r, \epsilon)$, then G contains a K_r^t.*

LEMMA 6. *Let (E_1, E_2) be a partition of $E(K_n)$ such that E_1 contains a C_m, where $m \geqslant 6$. Then*

(i) *if E_2 contains no K_r there is a cycle of length c, $m - 2r + 3 \leqslant c < m$, in E_1 (provided $m \geqslant 2r$),*

(ii) *if E_2 contains no C_r there is a cycle of length c', $m - 3 \leqslant c' < m$ in E_1 (provided $m \geqslant r$).*

Proof. Let $C = (x_1, ..., x_m)$ be a cycle of length m in G.

(i) Consider the vertices $x_1, x_3, ..., x_{2r-1}$. Since E_2 contains no K_r, some pair (x_i, x_j) of these vertices must be joined by an edge of E_1. Then E_1 contains the cycle $(x_1, x_2, ..., x_i, x_j, x_{j+1}, ..., x_m)$ of length at least $m - 2r + 3$.

(ii) Some (x_i, x_{i+2}), (x_i, x_{i+3}) or (x_i, x_{i+4}) must be in E_1, for otherwise it is easily seen that E_2 contains a C_r. It follows that E_1 contains a C_{m-3}, a C_{m-2}, or a C_{m-1}.

LEMMA 7. *Let (E_1, E_2) be a partition of $E(K_n)$ such that E_1 contains a cycle C of length m, but no C_{m+1}. If E_2 contains no K_r, then every vertex $x \notin V(C)$ is joined by edges of E_1 to at most $r - 1$ vertices of C.*

Proof. Let $C = (x_1, ..., x_m)$ and suppose that $x \notin V(C)$ is joined to vertices $x_{i_1}, ..., x_{i_r}$ of C (where $i_1 < i_2 < \cdots < i_r$). Then $(x_{i_j - 1}, x_{i_k - 1}) \in E_2$

for all j, k, $1 \leqslant j < k \leqslant r$, since otherwise E_1 would contain the $m + 1$-cycle

$$(x_1, ..., x_{i_j-1}, x_{i_k-1}, x_{i_k-2}, ..., x_{i_j}, x, x_{i_k}, x_{i_k+1}, ..., x_m).$$

But this contradicts the hypothesis that E_2 contain no K_r.

3. Main Results

THEOREM 1. $R(C_n, C_{2r-1}) = 2n - 1$ *if* $n > r(2r - 1)$.

Proof. By Lemma 1, $R(C_n, C_{2r-1}) \geqslant 2n - 1$. We prove the reverse inequality. Consider a partition (E_1, E_2) of $E(K_{2n-1})$ and assume that there is neither a C_n in E_1 nor a C_{2r-1} in E_2. It follows that, by Lemma 4, $| E_2 | \leqslant \frac{1}{4}(2n - 1)^2$, and hence that

$$| E_1 | \geqslant \binom{2n-1}{2} - \frac{1}{4}(2n - 1)^2.$$

But then, by Lemma 3, E_1 contains a cycle of length at least $n - 1$. By Lemma 6(ii), E_1 contains a cycle C of length $n - 2$ or $n - 1$. Let $S = V(K_{2n-1}) - V(C)$. Since $| S | \geqslant n$, there are vertices x_1, x_2 in S with the edge (x_1, x_2) in E_2. Choose further vertices $x_3, ..., x_r$ of S. Now, by Lemma 7, each x_i is joined by edges of E_1 to at most $2r - 2$ vertices of C. It follows that there are at least $n - 2 - r(2r - 2)$ vertices of C all of which are joined to each x_i by edges of E_2. But $n > r(2r - 1)$ by hypothesis. So E_2 contains a $K(r, r - 1)$ plus an additional edge, and this in turn contains a C_{2r-1}.

THEOREM 2. $2n - 1 \rightarrow (C_n, C_n)$ *if* $n > 3$.

Proof. Let (E_1, E_2) be a partition of $E(K_{2n-1})$ and suppose, without loss of generality, that $| E_1 | \geqslant | E_2 |$. Then

$$| E_1 | \geqslant \frac{1}{2}\binom{2n-1}{2}$$

and so, by Lemma 3, E_1 contains a cycle of length at least n.

We first show that if one of E_1 and E_2 contains a C_{2r+1} then one of E_1 and E_2 also contains a C_{2r} $(r > 2)$. For suppose that $(x_0, ..., x_{2r})$ is a C_{2r+1} in E_1 and that neither E_1 nor E_2 contains a C_{2r}. Then, taking indices modulo $2r + 1$,

$$\begin{aligned} & (x_i, x_{i+1}) \in E_1, \qquad 0 \leqslant i \leqslant 2r, \\ \Rightarrow\ & (x_i, x_{i+2}) \in E_2, \qquad 0 \leqslant i \leqslant 2r, \end{aligned}$$

since the $2r$-cycle $(x_0, x_1, \ldots, x_i, x_{i+2}, x_{i+3}, \ldots, x_{2r}) \notin E_1$,

$$\Rightarrow \quad (x_i, x_{i+4}) \in E_1, \qquad 0 \leqslant i \leqslant 2r,$$

since the $2r$-cycle $(x_i, x_{i+4}, x_{i+6}, \ldots, x_{i-2}) \notin E_2$,

$$\Rightarrow \quad (x_i, x_{i+3}) \in E_2, \qquad 0 \leqslant i \leqslant 2r,$$

since the $2r$-cycle $(x_i, x_{i+3}, x_{i+4}, \ldots, x_{i-2}, x_{i+2}, x_{i+1}) \notin E_1$. But then E_2 contains the $2r$-cycle

$$(x_{2r-1}, x_1, x_3, \ldots, x_{2r-5}, x_{2r-2}, x_{2r-4}, \ldots, x_2, x_{2r}, x_{2r-3}).$$

Now suppose that one of E_1 and E_2, say E_1, contains a $C_{2r}(2r > n)$ but that neither E_1 nor E_2 contains a C_{2r-1}. (Clearly if this is never the case then, by the above remarks, either E_1 or E_2 contains a C_n as desired.) Let $(x_1, \ldots, x_{2r})$ be this C_{2r}. Then, taking indices modulo $2r$,

$$(x_i, x_{i+1}) \in E_1, \qquad 1 \leqslant i \leqslant 2r,$$

and so, as before,

$$(x_i, x_{i+2}) \in E_2, \qquad 1 \leqslant i \leqslant 2r.$$

Moreover $(x_i, x_{i+2k}) \in E_2, 1 \leqslant i \leqslant 2r, 1 \leqslant k \leqslant r-1$. For if $(x_i, x_{i+2k}) \in E_1$, then

$$(x_{i-1}, x_{i+2k-2}) \in E_2,$$

since the $2r-1$-cycle

$$(x_i, x_{i+2k}, x_{i+2k+1}, \ldots, x_{i-1}, x_{i+2k-2}, x_{i+2k-3}, \ldots, x_{i+1}) \notin E_1,$$

and also

$$(x_{i+1}, x_{i+2k+2}) \in E_2,$$

since the $2r-1$-cycle

$$(x_i, x_{i+2k}, x_{i+2k-1}, \ldots, x_{i+1}, x_{i+2k+2}, x_{i+2k+3}, \ldots, x_{i-1}) \notin E_1.$$

But then E_2 contains the $2r-1$-cycle

$$(x_{i+1}, x_{i+3}, \ldots, x_{i-1}, x_{i+2k-2}, x_{i+2k-4}, \ldots, x_{i+2k+2}),$$

a contradiction.

We now have the following situation: the sets

$$X_1 = \{x_1, x_3, \ldots, x_{2r-1}\}, \qquad \text{and} \qquad X_2 = \{x_2, x_4, \ldots, x_{2r}\}$$

each span complete subgraphs in E_2. Every edge from X_1 to X_2 is in E_1, except that all but two edges incident with one vertex may be in E_2. If n is even then, since E_1 contains a $K(r-1, r)$ with $2r > n$, *a fortiori* E_1

contains a C_n ; so assume that n is odd. Now it is clear that no vertex in $V(K_{2n-1}) - X_1 - X_2$ can be joined to both a vertex of X_1 and a vertex of X_2 by edges of E_1, for then E_1 would contain a C_n. It follows that every vertex of $V(K_{2n-1}) - X_1 - X_2$ must be joined by edges of E_2 to all of X_1 or to all of X_2. Since there are $2(n - r) - 1$ vertices in $V(G) - X_1 - X_2$, at least $n - r$ of these vertices must be joined by edges of E_2 to every vertex of either X_1 or X_2, say X_1. But then E_2 contains a C_n, and the theorem is proved.

Together with Lemma 1 this implies the

COROLLARY. $R(C_n, C_n) = 2n - 1$, *if n is odd.*

THEOREM 3. $R(C_n, K_r^{t+1}) = t(n - 1) + r$, *if* $n > n_1(r, t)$.

Proof. By induction on t. We first prove that, for $n > n_1(r, 1)$,

$$R(C_n, K_r^2) = n + r - 1.$$

The method is similar to that of Theorem 1. By Lemma 2 it suffices to show that $R(C_n, K_r^2) \leqslant n + r - 1$. Let (E_1, E_2) be a partition of $E(K_{n+r-1})$, and assume that there is no C_n in E_1 and no K_r^2 in E_2. By Lemma 5, $|E_2| \leqslant \frac{1}{2}\epsilon(n + r - 1)^2$, for $n > n(2, r, \epsilon)$, and hence $|E_1| \geqslant \frac{1}{2}cn^2$, for some positive constant c and all $n > n(2, r, \epsilon)$. It follows from Lemma 3 that there is a cycle of length at least cn in E_1 and hence, by Lemma 6, a cycle C of length less than n but at least $c'n$, for some positive constant c'. Since there is no K_r^2 in E_2 there is no K_{2r} in E_2, and, applying Lemma 7, we find, when $c'n \geqslant 2r^2$, r vertices of $V(K_{n+r-1}) - V(C)$ joined by edges of E_2 to r vertices of C. Hence, putting

$$n_1(r, t) = \max\left(\frac{2r^2}{c'}, n(2, r, \epsilon)\right),$$

we obtain the desired contradiction.

Suppose the theorem is true for $t - 1$, and let (E_1, E_2) be a partition of $E(K_{t(n-1)+r})$. By the same argument, if there is no K_r^{t+1} in E_2, then there is a cycle of length less than n but greater than c_1n in E_1. By the induction hypothesis, there is a K_r^t in E_2, disjoint from this cycle. Applying Lemma 7, if $c_1n \geqslant tr((t + 1) r - 1) + r$, we find a K_r^{t+1} in E_2.

Theorem 3 can be strengthened to

$$R(C_n, K(r_1, ..., r_{t+1})) = t(n - 1) + r, \qquad \text{if} \qquad n > n_1'(r, t),$$

where $r_i = r$, $1 \leqslant i \leqslant t$, and $r_{t+1} = \epsilon(r, t)n$. We omit details.

It is worth noting that Theorem 3 does not hold for all $r \leqslant n$, even in the case $t = 1$. For $R(C_n, K_n^2) > 3(n - 1)$ as is seen by the graph $G(n - 1, n - 1, n - 1)$.

Using more care in the proof of Theorem 3 we obtain the

COROLLARY. $R(C_n, C_{2r}) = n + r - 1$, *if* $n > 4r^2 - r + 2$.

Proof. By Lemma 4, we can assume that $|E_2| \leqslant \frac{1}{4}(n + r - 1)^2$ and hence that

$$|E_1| \geqslant \binom{n+r-1}{2} - \frac{1}{4}(n + r - 1)^2.$$

It follows that, applying Lemma 3, there is a cycle of length at least $\frac{1}{2}(n + r - 3)$ in E_1 and therefore, by Lemma 6(ii), a cycle of length less than n and at least $\frac{1}{2}(n + r - 3)$ in E_1. By Lemma 7, if $\frac{1}{2}(n + r - 3) \geqslant r(2r - 1) + r$, that is, if $n > 4r^2 - r + 2$, there is a K_r^2 in E_2 and hence, *a fortiori*, a C_{2r} in E_2.

It has been observed by Gyárfás that $n + r - 1 \rightarrow (C_n, C_{2r})$ does not hold for all $2r < n$ when n is odd. In fact we see from $G(2r - 1, 2r - 1)$ that

$$4r - 2 \nrightarrow (C_n, C_{2r}), \qquad \text{if } n \text{ is odd.}$$

Note that, by Theorem 3,

$$R(C_n, K_r) = R(C_n, K_1^r) = (r - 1)(n - 1) + 1$$

if n is large enough. We now strengthen this.

THEOREM 4. $R(C_n, K_r) = (r - 1)(n - 1) + 1$ *if* $n \geqslant r^2 - 2$.

Proof. By induction on r. Trivially $R(C_n, K_2) = n$. Suppose the theorem is true for $r - 1$ and let (E_1, E_2) be a partition of $E(K_N)$, where $N = (r - 1)(n - 1) + 1$ and $n \geqslant r^2 - 2$, such that there is neither a C_n in E_1 nor a K_r in E_2. Then, by Turán's theorem [7],

$$|E_2| \leqslant \frac{N^2(r-2)}{2(r-1)}$$

and hence

$$|E_1| \geqslant \binom{N}{2} - \frac{N^2(r-2)}{2(r-1)} = \frac{N((r-1)(n-2)+1)}{2(r-1)}.$$

By Lemma 3, there is a cycle of length at least $n - 1$ in E_1. Since E_1 contains no C_n, by Lemma 6(i) there is a cycle C of length c, $n - 2r + 4 \leqslant c < n$, in E_1. Choose c so that it is as large as possible, subject to these bounds. Then $c \geqslant n - 2r + 4 > (r - 1)^2$. Since $c \leqslant n - 1$, $|V(K_N - C)| \geqslant (r - 2)(n - 1) + 1$ and so, by the induction hypothesis, $E_2 - C$ contains a K_{r-1}, with vertices $x_1, \ldots, x_{r-1}$. Clearly, because E_2 contains no K_r, each vertex of C must be joined by an edge of E_1 to at least one x_i. It follows that some x_i is joined by edges of E_1 to at least r vertices of C. But, by Lemma 7, this is impossible. The theorem follows.

For arbitrary n and r we have the following result:

THEOREM 5. $nr^2 \rightarrow (C_n, K_r)$.

Outline of proof. Let (E_1, E_2) be a partition of $E(K_{nr^2})$ and assume that E_1 contains no C_n and that E_2 contains no K_r. Let K be the largest complete subgraph in E_2, of order $p < r$. Then each vertex not in K must be joined by an edge of E_1 to at least one vertex of K. It follows that some vertex x of K is joined to a large set S (with $|S| = rn$) of vertices by edges of E_1. In the subgraph spanned by S, E_2 contains no K_r and so, by Turán's theorem [7], $|E_1| > \frac{1}{2}rn(n-1)$. By Lemma 3, E_1 contains a path of length $n-2$ in the subgraph spanned by S. This path, together with the edges from its end-vertices to $x \in V(K)$ gives us a C_n in E_1.

4. COMMENTS

We have not been able to evaluate $R(G_1, ..., G_k)$ for $k > 2$ even in the case of cycles. It is easy to see that, when $G_i \cong C_n$, $1 \leqslant i \leqslant k$, and n is odd,

$$R(G_1, ..., G_k) \geqslant 2^{k-1}(n-1) + 1.$$

On the other hand we can show that, in this case,

$$R(G_1, ..., G_k) \leqslant (k+2)!n.$$

Also of interest would be to find $R(C_n, C_r)$, $R(C_n, K_r)$, and $R(C_n, K_r^2)$ for all values of n and r. Since, by [4], $R(C_4, K_4) = 10$ it is possible that

$$R(C_n, K_4) = 3n - 2, \qquad \text{for all} \quad n > 3.$$

And $R(C_6, C_6) = 8$ leads to the conjecture that

$$R(C_{2n}, C_{2n}) = 3n - 1, \qquad \text{for all} \quad n > 2.$$

Note added in proof. There has been considerable development in the theory of Ramsey numbers since the writing of this paper. R. J. Faudree and R. H. Schelp [8] and, independently, V. Rosta [9], have shown that, except for $R(C_3, C_3)$ and $R(C_4, C_4)$,

$$R(C_m, C_n) = \begin{cases} 2n-1, \text{ for } 3 \leqslant m \leqslant n, m \text{ odd} \\ n + (m/2) - 1, 4 \leqslant m \leqslant n, m, n \text{ even} \\ \max\{n + (m/2) - 1, 2n - 1\}, 4 \leqslant m < n, m \text{ even}, n \text{ odd}. \end{cases}$$

Faudree and Schelp have also shown that

$$R(P_m, P_n) = n + [(m+1)/2] \text{ for } 1 \leqslant m \leqslant n,$$

and that

$$R(C_m, P_n) = \begin{cases} 2n+1, 3 \leqslant m \leqslant n, m \text{ odd}, \\ n + (m/2), 4 \leqslant m \leqslant n, m \text{ even}, \\ m + [(n+1)/2] - 1, 1 \leqslant n < m, m \text{ even} \geqslant 4, \\ \max\{m + [n + 1/2] - 1, 2n+1\}, 1 \leqslant n < m, m \text{ odd}, \end{cases}$$

where P_n is a path of length n. T. D. Parsons [10] has evaluated $R(C_4, P_n)$ and $R(K_m, P_n)$. ([8] R. J. Faudree and R. H. Schelp, All Ramsey numbers for cycles in graphs, submitted to Discrete Mathematics. [9] V. Rosta, submitted to J. Combinatorial Theory. [10] T. D. Parsons, personal communication.)

References

1. J. A. Bondy, Large cycles in graphs, *Discrete Mathematics* **1** (1971), 121–32.
2. G. Chartrand and S. Schuster, On the existence of specified cycles in complementary graphs, *Bull. Amer. Math. Soc.* **77** (1971), 995–8.
3. V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, II, Small diagonal numbers, *Proc. Amer. Math. Soc.* **32** (1972), 389–94.
4. V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, III, Small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.
5. P. Erdös and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337–56.
6. P. Erdös and A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946), 1087–91.
7. P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz. Lapok* **48** (1941), 436–52.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.