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Some remarks on Ramsey's and Turán's theorem

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1. In this paper we are going to discuss some special cases of a general problem which might be considered as being on the one hand a generalisation of the problem raised and solved by the well-known theorem of Turán, on the other hand as the well known problem of the Ramsey-numbers.

Before going to explain this in details, we give the notations we shall use:

$G(n)$ is a graph with n vertices

$G(n;e)$ is a graph with n vertices and e edges

$e(G)$ denotes the number of edges of G

$\bar{G}$ is the complementary graph of G

$K(\nu)$ is the complete graph with ν vertices

$H(n;k,\ell)$ is the class of $G(n)$ graphs, where $G(n)$ contains no $K(k)$ and $\bar{G}(n)$ contains no $K(\ell)$

$H(n;k)$ is the class of $G(n)$ graphs, where $G(n)$ contains no $K(k)$

$$f(n;k,\ell) \overset{\text{def}}{=} \begin{cases} \max\limits_{G \in H(n;k,\ell)} e(G) & \text{if} \quad H(n;k,\ell) \neq \emptyset \\ 0 & \text{if} \quad H(n;k,\ell) = \emptyset \end{cases}$$

$$f(n;k) \overset{\text{def}}{=} \max_{G \in H(n;k)} e(G)$$

$G(x_1, \dots, x_k)$ denotes the subgraph of G spanned by the vertices $x_1, \dots, x_k$.

The well-known, special form of Ramsey's theorem [5] asserts that for any k, ℓ there exists a $N(k,\ell)$ such that if $n > N(k,\ell)$ then $H(n;k,\ell) = \emptyset$.

The well-known theorem of Turán [6] gives the exact value of $f(n;k)$ namely that

$$f(n;k) = \frac{1}{2}\,\frac{k-2}{k-1}\,(n^2 - r^2) + \binom{r}{2} \quad \text{where} \quad n \equiv r \bmod (k-1) \qquad 0 \le r < k-1 .$$

The only "extreme graph" in $H(n;k)$ with $e = f(n;k)$ is the complete $k-1$ chromatic graph in each class having $\left[\frac{n}{k-1}\right]$ resp. $\left[\frac{n}{k-1}\right]+1$ vertices. It is worthy of note that for this graph $\bar{G}(n)$ contains a rather "large" complete graph (with $\left[\frac{n}{k-1}\right]$ vertices).

Now the general problem is to determine $f(n;k,\ell)$.

In the special - extremal-case when $\ell = n+1$ (i.e. if there is no condition on the complementary graph), $f(n;k,\ell) = f(n;k)$ is determined by Turán's theorem.

In the other special case, when k and ℓ are fixed and n is large enough, $f(n;k,\ell) = 0$ by Ramsey's theorem. The exact determination of $f(n,k,\ell)$ is probably hopeless, since this would imply the determination of the Ramsey-numbers. But one might expect - having in mind the remark in connection with Turán's theorem, - that $f(n;k,\ell)$ is essentially smaller, than $f(n;k)$ when ℓ is supposed to be much smaller than $\left[\frac{n}{k-1}\right]$.

It is easy to show that for every $c < 1$

(1) $$f(n; k, c\frac{n}{k-1}) < g(c)\,\frac{1}{2}\cdot\frac{k-2}{k-1}\,n^2$$

with $g(c) < 1$, but we cannot determine the exact value of $g(c)$. We do not prove (1) in this paper, but hope to return to it, and to other related questions, at another occasion.

2. In this paper we first investigate the case when k is fixed and $\ell = o(n)$.

Trivally $f(n;3,\ell) \le \frac{n\ell}{2}$* since if G contains no triangle and has a vertex of valency v, the v vertices joined to this vertex must be independent. Therefore $f(n;3,\ell) = o(n^2)$ if $\ell = o(n)$.

For the general case we prove

THEOREM 1. If $\ell = o(n)$ then

(4) $$f(n; 2r+1, \ell) = \frac{1}{2}(1-\frac{1}{r})\,n^2\,(1+o(1)) .$$

REMARK:

We cannot settle the case $k=4$. Perhaps

(3) $$f(n;4,\ell) = o(n^2)$$

if $\ell = o(n)$. We only get crude upper bounds for $f(n;4,\ell)$

If (3) holds, we can deduce for each fixed r and $\ell = o(n)$

(4) $$f(n; 2r+2, \ell) = \frac{1}{2}(1-\frac{1}{r})\,n^2\,(1+o(1)) .$$

Now we prove Theorem 1. First we prove it if $r=2$, i.e. we prove that if $\ell = o(n)$ then

*In some cases $f(n;3,\ell) = \frac{n\ell}{2}$. See [1], [2].

(5) $$f(n;5,\ell) = (1+o(1))\frac{n^2}{4}.$$

First we show that for sufficiently large n

(6) $$f(n;5,cn^{\frac{1}{2}}\log^2 n) > \frac{n^2}{4}.$$

It is well known [3] that there is a $G(m)$ which contains no triangle and for which $\bar{G}(m)$ contains no $K([cm^{1/2}\log^2 m])$.

Let $G_1([\frac{n}{2}])$ and $G_2([\frac{n+1}{2}])$ be two such graphs which do not have a common vertex. Join every vertex of G_1 to all the vertices of G_2. The resulting graph clearly proves (6).

To complete the proof of (5) we have to show that if $n > n_0(\varepsilon)$ and $G(n;[\frac{n^2}{4}(1+\varepsilon)])$ does not contain a $K(5)$ then $\bar{G}$ contains a $K([c_\varepsilon n])$ where c_ε depends only on ε.

First we show the following

LEMMA. Let $0<\alpha<\frac{1}{2}$ and $G(n;[\alpha n^2(1+\varepsilon)])$ be any graph. Then there is a subgraph $G(m)$, $m>c_{\varepsilon,\alpha}n$ each vertex of which has in $G(m)$ valency greater than $2\alpha m(1+\frac{\varepsilon}{4})$.

Let us assume that our Lemma is false. Then we can write the vertices in a sequence $x_1,\dots,x_n$ so that for every $k<(1-c)n$ the valency of x_k in $G(x_k,\dots,x_n)$ is less than $2\alpha(n-k)(1+\frac{\varepsilon}{2})$. But then

$$e(G(n)) = \left[\alpha n^2(1+\frac{\varepsilon}{4})\right] < 2\alpha(1+\frac{\varepsilon}{2})\sum_{k=0}^{n-1}(n-k)+\binom{[cn]}{2} <$$

$$< \alpha n^2(1+\frac{\varepsilon}{2})+\frac{c^2n^2}{4}$$

which is an evident contradiction if $c<\sqrt{\alpha\cdot\varepsilon}$.

Now we use the Lemma with $\alpha=\frac{1}{4}$. Let $G(m)$, $m>cn$

be a subgraph of our $G(n;[\frac{n^2}{4}(1+\varepsilon)]$ each vertex of which has valency $> \frac{m}{2}(1+\frac{\varepsilon}{4})$.

Let $G(x_1, x_2, x_3)$ be a triangle of our $G(m)$ (clearly every edge of $G(m)$ is contained in a triangle). Let $y_1, \dots, y_{m-3}$ be the other vertices of our $G(m)$. Each vertex of $G(m)$ has valency at least $\frac{m}{2}(1+\frac{\varepsilon}{2})$, hence more than $\frac{3}{2}m$ edges of type (x_i, y_j) $1 \le i \le 3$, $1 \le j \le m-3$ are in our $G(m)$.

Thus more than $\frac{m}{6}$ y_i's are joined to the same two x_i's say x_1 and x_2. If these y_i's are independent we have found an $K([cn])$ in $\bar{G}(m)$.

If y_r and y_s are joined, then $G(x_1, x_2, y_r, y_s)$ is a $K(4)$ in our $G(m)$. Henceforth we can thus assume that $G(m)$ contains a $K(4)$.

Let $G(z_1, z_2, z_3, z_4)$ be a $K(4)$ of our $G(m)$ and $\omega_1, \dots, \omega_{m-4}$ are the other vertices of it. At least $2m(1+\frac{\varepsilon}{2}) + o(1)$ edges of the form (z_i, ω_j) belong to $G(m)$ $(1 \le i \le 4,\ 1 \le j \le m-4)$. Thus by a simple computation there are at least $\frac{\varepsilon m}{100}$ vertices ω_j which are joined to the same three z_i's. These ω_j's must be independent since otherwise $G(m)$ contains a $K(5)$ and this completes the proof of (5).

Now we prove (2) for general r. First we show

$$f(n; 2r+1, e) > \frac{1}{2}\left(1 - \frac{1}{r}\right) n^2 \tag{7}$$

The proof follows the proof of (6).

Let G_i; $1 \le i \le r$ be graphs of $\left[\frac{n}{r}\right]$ vertices (with disjoint set of vertices) which contains no triangle, and where $\bar{G}_i$ contains no $K([cn^{1/2}\log^2 n])$.

Join every vertex of G_i to every vertex of G_j for every $1 \le i < j \le r$. The resulting graph proves (7).

To complete the proof of (2), assume that it holds for $2r-1$ and we prove it for $2r+1$. Thus we have to prove that every

$$G(n;[\tfrac{1}{2}(1-\tfrac{1}{r}+\varepsilon)n^2])$$

either contains a $K(2r+1)$ or $\bar{G}$ contains a $K([cn])$ where c depends only on ε and r. The proof will be very similar to that of (5). First of all, from our Lemma we obtain that we can assume that our $G(n)$ contains a subgraph $G(m)$ with $m > c_{\varepsilon,r}n$ each vertex of which has the valency $> m(1-\frac{1}{r}+\frac{\varepsilon}{2})$ Clearly for this $G(m)$

$$e(G(m)) > \frac{1}{2}(1-\frac{1}{r}+\frac{\varepsilon}{2})m^2.$$

Hence by our induction hypothesis we can assume that our $G(m)$ contains a $K(2r-1)$ whose vertices are $x_1,\dots,x_{2r-1}$.

Denote by $y_1,\dots,y_{m-2r+1}$ the other vertices of $G(m)$. At least

$$(2r-1)(1-\frac{1}{r}+\frac{\varepsilon}{2})m+O(1) \;>\; (2r-3)m+\frac{m}{r}$$

edges of type (x_i,y_j), $1\le i\le 2r-1$, $1\le j\le m-2r+1$ belong to $G(m)$.

Thus as in the proof of (5) we obtain that there are at least $c_1 m$ $(c_1 = c_1(r))$ vertices of $G(m)$ which are joined to the same $2r-2$ x_i's, since all these vertices cannot be independent, two of them must be joined, thus our $G(m)$ contains a $K(2r)$.

Let now $z_1,\dots,z_{2r}$ be the vertices of this $K(2r)$ and let $\omega_1,\dots,\omega_{m+2r}$ be the other vertices of $G(m)$. At least

$$2r(1-\frac{1}{r}-\frac{\varepsilon}{2})m+O(1) \;=\; (2r-2)m+\varepsilon rm+O(1)$$

of the edges (z_i,ω_j), $1\le i\le 2r$, $1\le j\le m-2r$ belongs to our $G(m)$. Hence by the same argument as used in the proof of (5) at least $c_{\varepsilon,r}m$ vertices w_i are joined to the same $2r-1$ z_i's. If two of these z_i's are joined, $G(m)$ contains a $K(2r+1)$, if no two of them are joined, $G(m)$ contains a $K([c_{\varepsilon,r}m])$ and since $m > c_1 n$ the proof of Theorem 1 is complete.

3. We remark that (6) is nearly best possible. In fact we prove

(8) $$f(n; 5, [cn^{1/2}]) < \frac{1}{8}(1+\varepsilon)n^2$$

for every c and ε if $n > n_0(\varepsilon, c)$.

Let $G(n; [\frac{1}{8}(1+\varepsilon)n^2])$ be any graph for which $\bar{G}$ does not contain a $K([cn^{1/2}])$. We will show that it must contain a $K(5)$. First of all, observe that by our Lemma it must contain a subgraph $G(m)$, $m > c_\varepsilon n$ each vertex of which has valency $> \frac{1}{4}(1+\frac{\varepsilon}{2})m$ and therefore

(9) $$e(G(m)) > \frac{1}{8}(1+\frac{\varepsilon}{2})m^2.$$

Secondly observe that

(10) $$f(n; 4, cn^{1/2}) = o(n^2).$$

Namely if (10) would be false, there would exist a $G(n; [\delta n^2])$ which contains no $K(4)$ and $\bar{G}$ contains no $K([cn^{1/2}])$. G clearly contains a vertex of valency $[2\delta n]$ i.e. G has a vertex x which is joined to $y_1, \dots, y_s$, $s \geq [2\delta n]$.

By a result of Graver and Jackel [4] $G(y_1, \dots, y_s)$ must either contain a triangle or $\bar{G}(y_1, \dots, y_s)$ contains a $K([c_1 n^{1/2}])$. Both assumptions clearly lead to a contradiction. Thus (10) is proved.

(9) and (10) clearly imply that $G(m)$ contains a $K(4)$ with vertices (x_1, x_2, x_3, x_4). Since each of the x_i's $(1 \leq i \leq 4)$ have valency $> \frac{1}{4}(1+\frac{\varepsilon}{2})m$, there clearly are $c\varepsilon m > c_1 \varepsilon n$ vertices $y_1, \dots, y_\ell$ $(\ell > c_1 \varepsilon m)$ which are joined to the same two x_i's say to x_1 and x_2. $G(y_1, \dots, y_\ell)$ cannot contain a $K([c\sqrt{n}])$ thus by [4] $G(y_1, \dots, y_\ell)$ contains a triangle, say $G(y_1, y_2, y_3)$ but then $G(x_1, x_2, y_1, y_2, y_3)$ is a $K(5)$ of our $G(n)$, which completes the proof of (8).

Perhaps

$$f(n; 5, [cn^{1/2}]) = o(n^2)$$

is true, but we could not prove it.

4. As to the case $k = 2r$, we prove that assuming $f(n;4,\ell) = o(n^2)$ for $\ell = o(n)$ we have for every fixed r

$$f(n; 2r+2, \ell) = \frac{1}{2}\left(1 - \frac{1}{r}\right) n^2 (1 + o(1)) \tag{11}$$

For the sake of simplicity we only prove (11) for $r = 2$.

The proof of the general case is the same, only slightly more complicated.

$f(n;6,\ell) > \frac{n^2}{4}$ is trivial, (it follows from $f(n;5,\ell) > \frac{n^2}{4}$).
Thus to prove (11) for $r = 2$ we only have to show that for every $\varepsilon > 0$ there is a $c_\varepsilon > 0$ so that for every $G(n;[\frac{n^2}{4}(1+\varepsilon)])$ which contains no $K(6)$ $\bar{G}$ contains a $K([c_\varepsilon n])$ (we of course assume $f(n;4,\ell) = o(n^2)$).

From Lemma it follows that our $G(n)$ has a subgraph $G(m)$ with $m > c_1 n$ so that every vertex of $G(m)$ has in $G(m)$ valency greater than $\frac{1}{2}(1+\frac{\varepsilon}{2})m$. Let x be any vertex of $G(m)$, denote by $S(x)$ the set of vertices of $G(m)$ joined to x.

We evidently have

$$|S(x) \cap S(y)| > \frac{\varepsilon m}{2}. \tag{12}$$

Put

$$M = \max |S(x) \cap S(y)|$$

where the maximum is taken over every two vertices x and y of $G(m)$ which are joined. By (12) we have $M > \frac{\varepsilon m}{2}$.

Assume that for x_1 and x_2 we have $|S(x_1) \cap S(x_2)| = M$ and let $y_1, \dots, y_M$ be the vertices of $G(m)$ joined to both x_1 and x_2. Our assumption $f(M; 4, \ell) = o(M^2)$ clearly implies

$$e(G(z_1, \dots, z_M)) = o(M^2). \tag{13}$$

To see (13), observe that $G(z_1, \dots, z_M)$ cannot contain a $K(4)$ thus if (13) would not hold, then $\bar{G}(z_1, \dots, z_M)$ would contain a $K([c_\varepsilon m])$, which is impossible.

From (13) it immediately follows that for all but $o(m) = o(M)$ vertices the valency (in $G(z_1, \dots, z_M)$) is $o(M)$. Hence there is a subgraph $G(z_1, \dots, z_N)$ of $G(z_1, \dots, z_M)$ with $N = (1 + o(1))M$ each vertex of which (in $G(z_1, \dots, z_N)$) has valency $o(N)$. Since $N > \frac{\varepsilon m}{4}$ we can assume that the vertices $z_1, \dots, z_N$ are not all independent, without loss of generality we can assume that z_1 and z_2 are joined.

Now we prove

$$|S(z_1) \cap S(z_2)| > M$$

and this contradiction will prove our assertion.

Let $y_1, \dots, y_s$ be the vertices of our $G(m)$ different from $z_1, \dots, z_N$. Clearly both z_1 and z_2 are joined to at least $\frac{1}{2}(1 + \frac{\varepsilon}{2})m + o(m)$ of the y_i's. Thus we evidently have

$$|S(z_1) \cap S(z_2)| > m(1 + \frac{\varepsilon}{2}) - s + o(m) = M + \frac{\varepsilon}{2} m + o(m).$$

This contradiction completes the proof of (11).

Incidentally it is easy to see that if $f(n; 4, \ell) \neq o(n^2)$ then $f(n; 6, \ell) > \frac{n^2}{4}(1 + \varepsilon)$ for infinitely many n and $\ell = o(n)$.

To see this let G_1 and G_2 both have n vertices, every vertex of G_1 is joined to every vertex of G_2, G_1 contains no triangles, G_2 no $K(4)$, G_2 has more than εn^2 edges and both $\bar{G}_1$ and $\bar{G}_2$ do not contain a $K(\ell)$.

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