Partitions into primes.

Dedicated to the memory of Tibor Szele.

By P. T. BATEMAN in Urbana, Illinois and P. ERDŐS in Notre Dame, Indiana.

1. Introduction. Let $P(n)$ denote the number of partitions of the integer n into primes (1 is not counted as prime), repetitions being allowed. That is, $P(n)$ is the number of ways n can be expressed in the form $n_1 p_1 + n_2 p_2 + \ldots$, where p_j denotes the jth prime number and $n_1, n_2, \ldots$ are arbitrary non-negative integers. The purpose of this note is to prove that

$$P(n+1) \geqq P(n) \qquad (n=1,2,3,\ldots). \tag{1}$$

In another paper[1]) we have proved that, if A is any non-empty set of positive integers and $F_A(n)$ denotes the number of partitions of the integer n into parts taken from the set A, repetitions being allowed, then $F_A(n)$ is a non-decreasing function of n for *large*[2]) positive n if and only if either (I) A contains the element 1 or (II) A contains more than one element and, if we remove any single element from A, the remaining elements have greatest common divisor 1. This result shows that (1) is true if n is sufficiently large, and other results in the same paper show that in fact $\lim_{n\to\infty}\{P(n+1)-P(n)\} = +\infty$. However, the methods employed there do not provide a good estimate of the point at which the monotonicity of $P(n)$ begins. In the present paper we prove (1) by using an argument particularly adapted to the case where A is the set of prime numbers.

Let $P_k(n)$ denote the number of partitions of the integer n into parts taken from the first k primes, repetitions being allowed. Thus we have the formal power-series relation

$$\sum_{n=0}^{\infty} P_k(n) X^n = \prod_{j=1}^{k} (1-X^{p_j})^{-1}. \tag{2}$$

Since $P(n) = P_k(n)$ for $n < p_{k+1}$, the assertion (1) will be proved if we can establish the following by induction on k.

---

1) *Monotonicity of partition functions,* to be published in *Mathematika.*

2) In case (I) obviously $F_A(n+1) \geqq F_A(n)$ for all n.

If k is a positive integer greater than 2, then

(A_k) $P_k(n+1) \geqq P_k(n)$ *for any positive integer n, and*

(B_k) $P_k(p+1) > P_k(p)$ *for any prime number k greater than p_k.*

2. The case $k=3$. In view of (2) we have the formal power-series relation

$$1+\sum_{n=0}^{\infty}\{P_3(n+1)-P_3(n)\}\,X^{n+1}=\frac{1-X}{(1-X^2)(1-X^3)(1-X^5)}=$$
$$=\frac{1}{30(1-X)^2}+\frac{a}{1-X}+\sum_{\mu=1}^{2}\frac{b_\mu}{1-e^{2\pi i\mu/3}X}+\sum_{\nu=1}^{4}\frac{c_\nu}{1-e^{2\pi i\nu/5}X},$$

where $a, b_1, b_2, c_1, c_2, c_3, c_4$ are certain complex numbers which could be calculated but whose values we shall not require. Hence if n is a positive integer

$$P_3(n+1)-P_3(n)=\frac{n+2}{30}+a+\sum_{\mu=1}^{2}b_\mu e^{2\pi i\mu(n+1)/3}+\sum_{\nu=1}^{4}c_\nu e^{2\pi i\nu(n+1)/5},$$

and so

$$P_3(n+1)-P_3(n)=\left[\frac{n-1}{30}\right]+\psi(n), \tag{3}$$

where $[x]$ denotes the greatest integer not exceeding the real number x and ψ is a function on the positive integers which has period 30. The values of $\psi(n)$ can be most easily found by taking $n=1, 2, \ldots, 30$ in (3). We find that

$$\psi(n)=\begin{cases}0 & \text{if } n\equiv 0,\ 2,\ 3,\ 5,\ 6,\ 8,\ 10,\ 12,\ 15,\ 18,\ 20 \pmod{30},\\ 2 & \text{if } n\equiv 19,\ 29 \pmod{30},\\ 1 & \text{otherwise.}\end{cases}$$

Thus $\psi(n)\geqq 0$ for all n and $\psi(n)\geqq 1$ if $(n,30)=1$. Hence assertions (A_3) and (B_3) are valid.

3. Inductive step. Suppose k is a positive integer greater than 3 and assume that assertions (A_{k-1}) and (B_{k-1}) are valid. Then we shall show that (A_k) and (B_k) are valid.

We begin by remarking that if n is any integer

$$\begin{aligned} P_k(n) &= P_{k-1}(n)+P_k(n-p_k)= \\ &= P_{k-1}(n)+\begin{cases}0 & \text{if } n<p_k \text{ or } n=p_k+1,\\ 1 & \text{if } n=p_k,\\ P_k(n-p_k) & \text{if } n\geqq p_k+2.\end{cases}\end{aligned} \tag{4}$$

This is equivalent to the formal power-series identity

$$(1-X^{p_k})\sum_{n=0}^{\infty}P_k(n)\,X^n=\sum_{n=0}^{\infty}P_{k-1}(n)\,X^n,$$

which is an immediate consequence of (2). Alternatively (4) can be established by noticing that the first term on the right is equal to the number of partitions of n into parts taken from the first k primes in which p_k does not actually occur as a part, while the second term is equal to the number of partitions of n into parts taken from the first k primes in which p_k does actually occur as a part.

Now if $1 \leqq n < p_k$, then $P_k(n+1) \geqq P_{k-1}(n+1) \geqq P_{k-1}(n) = P_k(n)$ by (A_{k-1}) and (4). If $n = p_k$, then $P_k(n+1) = P_k(p_k+1) = P_{k-1}(p_k+1) \geqq$ $\geqq P_{k-1}(p_k)+1 = P_k(p_k) = P_k(n)$ by (B_{k-1}) and (4). If $n > p_k$ and if we have proved that $P_k(m+1) \geqq P_k(m)$ for $m = 1, 2, \ldots, n-1$, then

$$P_k(n+1) = P_{k-1}(n+1) + P_k(n+1-p_k) \geqq P_{k-1}(n) + P_k(n-p_k) = P_k(n)$$

by (A_{k-1}) and (4). Hence $P_k(n+1) \geqq P_k(n)$ for all positive integers n and so assertion (A_k) is proved.

Now suppose p is a prime number greater than p_k. Then

$$P_{k-1}(p+1) > P_{k-1}(p) \quad \text{by} \quad (B_{k-1}) \quad \text{and} \quad P_k(p+1-p_k) \geqq P_k(p-p_k)$$

by (A_k). Hence by (4)

$$P_k(p+1) = P_{k-1}(p+1) + P_k(p+1-p_k) > P_{k-1}(p) + P_k(p-p_k) = P_k(p).$$

Thus (B_k) is established and our proof is complete.

(Received September 5, 1955.)