Decidable versions of first order logic and cylindric-relativized set algebras

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Abstract

We investigate three technically equivalent subjects. (i) Algebras whose elements are relations of higher ranks (e.g. binary, ternary, n-ary etc.), and whose operations are the natural counterparts of logical connectives like e.g. quantification or substitution. (ii) Fine tuning first order logic via its model theory to achieve desirable properties ranging from decidability to well-behaved finite model theory, or 'tractable' Beth definability and Craig interpolation properties. (iii) A proof theoretic approach to (ii). The extra-Boolean operations on our algebras of relations correspond to quantification, substitution of individual variables, and the logical constant equality.

Elsewhere this work is applied to multi-dimensional modal logics, arrow logics, and to a new generation of generalized quantifiers.

A widely applicable new method called Mosaic Method, which has common roots with Tableaux Methods, is developed.

1 Introduction

We will investigate some properties of first order logic both from a model theoretic and from a proof theoretic perspective. Some of the results will be relevant to the connections between modal logic and first order logic and also to the theory of generalized quantifiers, cf. e.g. [27]. There will be results relevant to the theory of schemata of first order formulas cf. e.g. Rybakov [21], [19].

Our methods will yield results about algebraic logic, too. We will prove for various classes of algebras of relations (of higher rank) that their equational theories are decidable. E.g. we will prove that the class of relativized representable cylindric algebras has a decidable equational theory. We will look at various

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nice and useful subclasses of this class and will prove that their equational theories are also decidable. We will look at other properties of agebras as well as to other kinds of algebras.

Let us turn to the "purely logical" aspects of this paper. They are related to the relatively recent trend in logic known as arrow logic (cf. e.g. van Benthem [25], [26], [13], [12]), but familiarity with arrow logic is not necessary for understanding the present paper.

The recent paper van Benthem [25] asks the following question (see Appendix 2D therein): "What would have to be weakened in standard predicate logic to get an arrow-based decidable version?" Here we show that it is the permutability of quantifiers which is responsible for the undecidability of first order logic. More precisely: van Benthem's question can be understood in at least two different ways, depending on whether one has the Amsterdam or the Budapest "manifestation" of arrow logic in mind. Roughly speaking the difference is whether one choses a syntactic (or proof-theoretic) or a semantic (or model-theoretic) approach[1].

1. First we give a solution of the proof-theoretic (or Amsterdam way) version of the problem. Below we shall work with the restricted version of first order logic, which is well-known to be equivalent with the ordinary formulation, see [8] 4.3. Here restricted means that relational atomic formulas all have the form

$$R(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$$

where n is the arity of the relation symbol R. We also assume that there are no constant or function symbols in our languages (that this again is not a real restricton is well known, see e.g. the textbook [7]).

Consider the following inference system for restricted first order logic. The axioms are

((1)) φ, where φ is a (propositional) tautology
((2)) $\forall\mathbf{v}_i(\varphi \to \psi) \to (\forall\mathbf{v}_i\varphi \to \forall\mathbf{v}_i\psi)$
((3)) $\forall\mathbf{v}_i\varphi \to \varphi$
((4a)) $\forall\mathbf{v}_i\forall\mathbf{v}_j\varphi \to \forall\mathbf{v}_j\forall\mathbf{v}_i\varphi$
((4b)) $\forall\mathbf{v}_k\varphi \to \forall\mathbf{v}_k\forall\mathbf{v}_k\varphi$
((4c)) $\exists\mathbf{v}_k\varphi \to \forall\mathbf{v}_k\exists\mathbf{v}_k\varphi$
((4d)) $R(\bar{x}) \to \forall\mathbf{v}_k R(\bar{x})$ provided $\mathbf{v}_k \notin \mathrm{Rng}\bar{x}$ and $R(\bar{x})$ is an atomic formula,
((5)) $\mathbf{v}_i = \mathbf{v}_i$
((6)) $\exists\mathbf{v}_i(\mathbf{v}_i = \mathbf{v}_j)$
((7)) $\mathbf{v}_i = \mathbf{v}_j \to (\mathbf{v}_i = \mathbf{v}_k \to \mathbf{v}_j = \mathbf{v}_k)$
((8)) $\mathbf{v}_i = \mathbf{v}_j \to [\varphi \to \forall\mathbf{v}_i(\mathbf{v}_i = \mathbf{v}_j \to \varphi)]$ if $i \neq j$
((9)) $\exists\mathbf{v}_i\varphi \leftrightarrow \neg\forall\mathbf{v}_i\neg\varphi$

and the inference rules are Modus Ponens

[1] For definitions and notation we refer the reader to the subsequent sections (esp. section 2) of this paper.

(MP) if $\vdash \varphi, \varphi \rightarrow \psi$, then $\vdash \psi$
and Generalization
(G) if $\vdash \varphi$, then $\vdash \forall \mathbf{v}_i \varphi$.
This inference system is sound and complete for restricted first order logic (see [8] 4.3.23). Our first answer to van Benthem's question is the following theorem:

THEOREM 1.1 *The set of formulas derivable from ((1))...((3)), ((4b))...((4d)), ((5))...((9)) by (MP) and (G) is decidable.*

Proof: See Corollary 3.1 in section 3. ■

Thus while in arrow logic the way to get rid of undecidability is to leave out (or at least weaken) associativity, here we have to leave out ((4a)) (i.e. commutativity of quantifiers). We note that it is well known that associativity in relation algebras corresponds to commutativity of cylindrifications in cylindric algebras.

2. Let us turn to the model-theoretic (or Budapest) approach now. Here we give two modifications of ordinary first order semantics both of which make the set of valid formulas decidable, thus giving two answers for van Benthem's question.

Theorem 1.2(i) below says that the strength of first order logic disappears if we weaken the notion of validity by permitting certain "nonstandard" models. Theorem 1.2(ii) is connected with usual investigations in logic: we define (syntactically) a decidable set of formulas and prove that the usual validity is decidable for the elements of this set.

In the sequel $\mathrm{t} : \underline{R} \longrightarrow \omega$ is a relational type (so it does not contain function symbols) and $\mathrm{F_t}$ is the set of usual first order (not just restricted) formulas of type t. We use the set $V = \{\mathbf{v}_i : i \in \omega\}$ of variable symbols. $\mathrm{Mod_t}$ denotes the class of models of type t^2. For $\underline{M} \in \mathrm{Mod_t}$ $\underline{M} = \langle M, R^{\underline{M}} \rangle_{R \in \underline{R}}$, that is, M denotes the universe of $\underline{M}$ and $R^{\underline{M}} \subseteq {}^{\mathrm{t}(R)}M$ is the relation corresponding to R in $\underline{M}$.

DEFINITION 1.1 (i) Let $K \subseteq \mathrm{Mod_t}$. We say that K is a generalized Kripke-model or a partial model, and write $K \in \mathcal{K}_\mathrm{t}$, if

$$(\forall \underline{M}, \underline{N} \in K) \underline{M} \restriction (M \cap N) = \underline{N} \restriction (M \cap N),$$

that is, if K is a class of "compatible" models. We define the notion of validity of usual first order formulas in members of $\mathcal{K}_\mathrm{t}$:

Let $K \in \mathcal{K}_\mathrm{t}$. Then $\mathrm{Val}(K) \stackrel{\mathrm{def}}{=} \bigcup \{{}^\omega M : \underline{M} \in K\}$ is the set of possible valuations of the variable symbols in K. Let $h \in \mathrm{Val}(K)$, $R \in \underline{R}$, $\mathrm{t}(R) = n$,

[2] Thus, using the notation of section 2, $\mathrm{F_t} = \mathrm{Fm}_\omega^{\langle \omega, \mathrm{t} \rangle}$, and $\mathrm{Mod_t} = \mathrm{Mod(t)}$.

$i_1 \dots i_n, i, j \in \omega$ and $\varphi, \psi \in \mathrm{F_t}$. Then

$$
\begin{aligned}
&K \overset{\mathrm{k}}{\models} R(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_n})[h] \overset{\text{def}}{\iff} \langle h(i_1), \dots h(i_n)\rangle \in R^{\underline{M}} \text{ for some } \underline{M} \in K \\
&K \overset{\mathrm{k}}{\models} \mathbf{v}_i = \mathbf{v}_j[h] \overset{\text{def}}{\iff} h(i) = h(j) \\
&K \overset{\mathrm{k}}{\models} \exists \mathbf{v}_i \varphi[h] \overset{\text{def}}{\iff} K \overset{\mathrm{k}}{\models} \varphi[h(i/u)] \text{ for some } u \text{ such that } h(i/u) \in \mathrm{Val}(K) \\
&K \overset{\mathrm{k}}{\models} (\varphi \wedge \psi)[h] \overset{\text{def}}{\iff} (K \overset{\mathrm{k}}{\models} \varphi[h] \quad \text{and} \quad K \overset{\mathrm{k}}{\models} \psi[h]) \\
&K \overset{\mathrm{k}}{\models} \neg\varphi[h] \overset{\text{def}}{\iff} K \overset{\mathrm{k}}{\not\models} \varphi[h] \\
&K \overset{\mathrm{k}}{\models} \varphi \overset{\text{def}}{\iff} (\forall h \in \mathrm{Val}(K)) K \overset{\mathrm{k}}{\models} \varphi[h] \\
&\overset{\mathrm{k}}{\models} \varphi \overset{\text{def}}{\iff} (\forall K \in \mathcal{K}_{\mathrm{t}}) K \overset{\mathrm{k}}{\models} \varphi.
\end{aligned}
$$

(ii) Models with a prescribed set of valuations. Let

$$\mathcal{M}_{\mathrm{t}} \overset{\text{def}}{=} \{\langle \underline{M}, V\rangle : \underline{M} \in \mathrm{Mod_t} \quad \text{and} \quad V \subseteq {}^{\omega}M\}.$$

Let $\underline{A} = \langle \underline{M}, V\rangle \in \mathcal{M}_{\mathrm{t}}$, $k \in V$ and $R, i_1 \dots i_n, i, j, \varphi, \psi$ as in (i). Then

$$
\begin{aligned}
&\underline{A} \overset{\mathrm{m}}{\models} R(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_n})[k] \overset{\text{def}}{\iff} \langle k(i_1), \dots k(i_n)\rangle \in R^{\underline{M}} \\
&\underline{A} \overset{\mathrm{m}}{\models} \mathbf{v}_i = \mathbf{v}_j[k] \overset{\text{def}}{\iff} k(i) = k(j) \\
&\underline{A} \overset{\mathrm{m}}{\models} \exists \mathbf{v}_i \varphi[k] \overset{\text{def}}{\iff} (\exists u)[\underline{A} \overset{\mathrm{m}}{\models} \varphi[k(i/u)] \quad \text{and} \quad k(i/u) \in V] \\
&\underline{A} \overset{\mathrm{m}}{\models} (\varphi \wedge \psi)[k] \overset{\text{def}}{\iff} (\underline{A} \overset{\mathrm{m}}{\models} \varphi[k] \quad \text{and} \quad \underline{A} \overset{\mathrm{m}}{\models} \psi[k]) \\
&\underline{A} \overset{\mathrm{m}}{\models} \neg\varphi[k] \overset{\text{def}}{\iff} \underline{A} \overset{\mathrm{m}}{\not\models} \varphi[k] \\
&\underline{A} \overset{\mathrm{m}}{\models} \varphi \overset{\text{def}}{\iff} (\forall k \in V) \underline{A} \overset{\mathrm{m}}{\models} \varphi[k] \\
&\overset{\mathrm{m}}{\models} \varphi \overset{\text{def}}{\iff} (\forall \underline{A} \in \mathcal{M}_{\mathrm{t}}) \underline{A} \overset{\mathrm{m}}{\models} \varphi.
\end{aligned}
$$

(iii) ρ is an atomic formula if it is of the form $R(\mathbf{v}_{i_0}, \dots, \mathbf{v}_{i_{n-1}})$ where $R \in \underline{R}$, $n = \mathrm{t}(R)$ and $i \in {}^{n}\omega$.

$\varphi \in \mathrm{F_t}$ is said to be relativized if there is an atomic formula ρ such that φ is of the form $\rho \to \psi$ where ψ is built up from atomic formulas using $\neg$, $\wedge$ and "$\exists \mathbf{v}_i(\rho \wedge \dots)$", and all variable symbols occurring in ψ occur in ρ. More precisely:

Let $R \in \underline{R}$, $n = \mathrm{t}(R)$, $i_1, \dots, i_n \in \omega$ and $\rho = R(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_n})$. Then $\mathrm{RL}(\rho) \subseteq \mathrm{F_t}$ is the smallest set satisfying

(i) $\eta \in \mathrm{RL}(\rho)$ if η is an atomic formula such that all variable symbols occurring in η are among $\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_n}$

(ii) $\{\neg\eta, \eta \wedge \xi, \exists \mathbf{v}_{i_k}(\rho \wedge \eta)\} \subseteq \mathrm{RL}(\rho)$ if $\eta, \xi \in \mathrm{RL}(\rho)$ and $1 \leq k \leq n$.

$$\mathrm{RF_t} \stackrel{\mathrm{def}}{=} \{\rho \to \psi : \rho \in \mathrm{F_t} \quad \text{is atomic and} \quad \psi \in \mathrm{RL}(\rho)\}.$$

We say that φ is relativized if $\varphi \in \mathrm{RF_t}$.

Finally, φ is an ordinary relativized formula, $\varphi \in \mathrm{SRF_t}$, if φ is built up from elements of $\{\rho \wedge \eta : \eta \quad \text{is atomic}\}$ using $\wedge$, $\rho \wedge \neg$ and $\rho \wedge \exists \mathrm{v}_i$, and all variable symbols occurring in φ occur in ρ. ■

We note that $\stackrel{\mathrm{k}}{\models}$ and $\stackrel{\mathrm{m}}{\models}$ are generalizations of $\models$. If $K \in \mathcal{K}_\mathrm{t}$ is a singleton, then $(\forall\varphi)[K \stackrel{\mathrm{k}}{\models} \varphi \Leftrightarrow K \models \varphi]$, and if $\langle \underline{M}, V\rangle \in \mathcal{M}_\mathrm{t}$ is such that $V = {}^\omega M$ then $(\forall\varphi)[\langle \underline{M}, V\rangle \stackrel{\mathrm{m}}{\models} \varphi \Leftrightarrow \underline{M} \models \varphi]$.

THEOREM 1.2 (i) $\{\varphi \in \mathrm{F_t} : \stackrel{\mathrm{k}}{\models} \varphi\}$ *is decidable, that is, for every formula it is decidable whether it is valid in generalized Kripke-models. Similarly,* $\{\varphi \in \mathrm{F_t} : \stackrel{\mathrm{m}}{\models} \varphi\}$ *is decidable.*

(ii) $\{\varphi \in \mathrm{RF_t} : \models \varphi\}$ *and* $\{\varphi \in \mathrm{SRF_t} : \not\models \neg\varphi\}$ *is decidable, that is, validity is decidable for relativized formulas and satisfiability is decidable for ordinary relativized formulas.*

Proof: See Corollary 4.1 in section 4. ■

REMARK 1.1 (i) $\langle \mathrm{F_t}, \mathcal{K}_\mathrm{t}, \stackrel{\mathrm{k}}{\models}\rangle$ and $\langle \mathrm{F_t}, \mathcal{M}_\mathrm{t}, \stackrel{\mathrm{m}}{\models}\rangle$ are the logics corresponding to the classes of algebras G_ω and Crs_ω (to be defined later), respectively, in the sense of [5] or [8] 5.6.

(ii) Validity for members of $\mathrm{SRF_t}$ is also decidable in the obvious way: if $\varphi \in \mathrm{SRF_t}$ then φ is not valid since it is of the form $\rho \wedge \psi$ where ρ is atomic. Deciding the satisfiability of members of $\mathrm{SRF_t}$ is far from being trivial. ■

2 Definitions, notation

Here we define only those notions and notations which are not universally adopted in the literature.

2.1 Set-theoretic notions

Throughout, we use the von Neumann ordinals. The smallest infinite ordinal (the set of natural numbers) is denoted by ω.

- SbA is the powerset (set of <u>s</u>u<u>b</u>sets) of A
- $A \subset B \stackrel{\mathrm{def}}{\Longleftrightarrow} (A \subseteq B \quad \text{and} \quad A \neq B)$

- $A \subseteq_\omega B \overset{\text{def}}{\iff} (A \subseteq B \quad \text{and} \quad |A| < \omega)$
- (a, b) as well as $\langle a, b\rangle$ denote the pair of a and b.

Let $R, S \subseteq A \times B$ be relations and H a set. Then

- $\text{Dom}R \overset{\text{def}}{=} \{a \in A : \exists b(a, b) \in R\}$, the domain of R
- $\text{Rng}R \overset{\text{def}}{=} \{b \in B : \exists a(a, b) \in R\}$, the range of R
- $R^\star H \overset{\text{def}}{=} \{b \in B : (\exists a \in H)(a, b) \in R\}$, the R-image of H
- $R \restriction H \overset{\text{def}}{=} \{(a, b) \in R : a \in H\}$, the restriction of R to H
- $R \mid S \overset{\text{def}}{=} \{(a, b) : \exists c[(a, c) \in R \quad \text{and} \quad (c, b) \in S]\}$, the composition of R and S
- $R^{-1} \overset{\text{def}}{=} \{(b, a) : (a, b) \in R\}$, the inverse of R.

Id $\overset{\text{def}}{=} \{(H, H) : H \quad \text{is a set}\}$ denotes the identity class, and if H is a set then $\text{Id}_H \overset{\text{def}}{=} \{(u, u) : u \in H\}$ is the identity relation on H.

Now we list notations used in connection with functions.

- $f : A \longrightarrow B \overset{\text{def}}{\iff} f$ is a function mapping A into B
- $f : A \rightarrowtail B \overset{\text{def}}{\iff} f$ is one-one
- $f : A \twoheadrightarrow B \overset{\text{def}}{\iff} f$ is onto
- $f : A \rightarrowtail\!\!\!\twoheadrightarrow B \overset{\text{def}}{\iff} f$ is one-one and onto
- ${}^AB \overset{\text{def}}{=} \{f : f : A \longrightarrow B\}$
- $\ker f \overset{\text{def}}{=} \{(a, b) \in \text{Dom}f \times \text{Dom}f : f(a) = f(b)\}$, the kernel of f
- $f \circ g \overset{\text{def}}{=} g \mid f$
- $f(x/u) \overset{\text{def}}{=} f^x_u \overset{\text{def}}{=} (f \setminus \{\langle x, f(x)\rangle\}) \cup \{\langle x, u\rangle\}$.

f is a sequence if f is a function with domain α for some ordinal α. We do not distinguish 2-sequences (i.e. functions with domain 2) from pairs.

2.2 Algebraic notions

t' is a (mixed) similarity type (or simply type) if $t' = (t, F)$, where $t : \underline{R} \longrightarrow \omega \setminus 1$ for some $\underline{R}$, and $F \subseteq \underline{R}$. If $R \in \underline{R} \setminus F$ then R is a relation symbol of arity $t(R)$; if $R \in F$ then R is a function symbol of arity $t(R) - 1$. Zero-ary function symbols are called constant symbols.

Let $t' = (t, F)$ be a type, $t : \underline{R} \longrightarrow \omega \setminus 1$. $\underline{A}$ is a structure of type t' if $\underline{A} = \langle A, \mathbb{F} \rangle$, where $\mathbb{F}$ is a function with $\mathrm{Dom}\mathbb{F} = \mathrm{Dom}{\approx}$. $R^{\underline{A}} \stackrel{\text{def}}{=} \mathbb{F}(\mathbb{R})$ is a relation of arity $t(R)$ if $R \in \underline{R} \setminus F$ and R is a $t(R) - 1$-place function otherwise.

t is a relational (resp. algebraic) (similarity) type if $t' = (t, 0)$ (resp. $t' = (t, \mathrm{Dom}t)$) is a mixed similarity type; and $\underline{A}$ is a model (resp. algebra) of type t if $\underline{A}$ is a structure of type t'. $\mathrm{Mod}(t)$ (resp. $\mathsf{Alg}(t)$) denotes the class of models (resp. algebras) of type t.

Let H be a set and let $\underline{A}, \underline{B} \in \mathrm{Mod}(t')$ for some type t'. Then

$$\underline{A} \restriction H \stackrel{\text{def}}{=} \langle A \cap H, R^{\underline{A}} \cap {}^{t(R)}H \rangle_{R \in \underline{R}}$$

is the restriction of $\underline{A}$ to H. Note that $\underline{A} \restriction H$ is not necessarily a structure of type t' but it is always a model of type t.

- $\underline{A} \subseteq \underline{B} \stackrel{\text{def}}{\Longleftrightarrow} \underline{A} = \underline{B} \restriction A$ ($\underline{A}$ is a submodel of $\underline{B}$)
- $f : \underline{A} \longrightarrow \underline{B} \stackrel{\text{def}}{\Longleftrightarrow} f : A \longrightarrow B$ and $\{f \circ s : s \in R^{\underline{A}}\} \subseteq R^{\underline{B}}$ (f is a homomorphism from $\underline{A}$ to $\underline{B}$)
- $f : \underline{A} \rightarrowtail \underline{B} \stackrel{\text{def}}{\Longleftrightarrow} f : \underline{A} \longrightarrow B$ and $f : A \rightarrowtail B$
- $f : \underline{A} \twoheadrightarrow \underline{B} \stackrel{\text{def}}{\Longleftrightarrow} f : \underline{A} \longrightarrow B$ and $f : A \twoheadrightarrow B$
- $f : \underline{A} \rightarrowtail\!\!\!\twoheadrightarrow \underline{B} \stackrel{\text{def}}{\Longleftrightarrow} f : \underline{A} \longrightarrow B$ and $f : A \rightarrowtail\!\!\!\twoheadrightarrow B$.

From now on let t be an algebraic type, $\underline{A}, \underline{B} \in \mathsf{Alg}(t)$ and $K \subseteq \mathsf{Alg}(t)$.

- $\mathrm{Sg}^{\underline{A}} H \stackrel{\text{def}}{=} \bigcap \{Y \subseteq A : \underline{A} \restriction Y$ is an algebra, and $H \subseteq Y\}$
- $\underline{Sg}^{\underline{A}} H \stackrel{\text{def}}{=} \underline{A} \restriction \mathrm{Sg}^{\underline{A}} H$
- $\mathbf{I}K$, $\mathbf{H}K$, $\mathbf{S}K$ and $\mathbf{P}K$ denotes the class of isomorphic copies, homomorphic images, subalgebras and direct products of elements of K respectively
- $\mathrm{Tm}_X(t)$ denotes the set of terms of type t built up using elements of X (as variable symbols)
- $\underline{\mathrm{Tm}}_X(t)$ denotes the natural t-type algebra with universe $\mathrm{Tm}_X(t)$
- $\tau^{\underline{A}}[k]$ denotes the value of the term τ in the algebra $\underline{A}$ under the valuation k
- $\underline{\mathcal{F}}_X K \stackrel{\text{def}}{=} \underline{\mathrm{Tm}}_X(t)/\mathrm{Cr}_X K$, where $\mathrm{Cr}_X K \stackrel{\text{def}}{=} \bigcap \{R : \underline{\mathrm{Tm}}_X(t)/R \in \mathbf{IS}K\}$.

2.3 First order languages

Λ is a language if $\Lambda = \langle \alpha, t\rangle$, where α is an ordinal and t is a relational type. Throughout, $\mathbf{v}$ is a unique class-function with domain the class of all ordinals. In all our languages we shall use the $\mathbf{v}_i$'s as variable symbols. For easier readability we define $x \stackrel{\text{def}}{=} \mathbf{v}_0$, $y \stackrel{\text{def}}{=} \mathbf{v}_1$ and $z \stackrel{\text{def}}{=} \mathbf{v}_2$. We assume that the usual disjointness conditions are satisfied (e.g. $\mathrm{Rng}\mathbf{v} \cap \mathrm{Dom}t = 0$).

Let $\Lambda = \langle \alpha, t\rangle$ be a language and let $\underline{R} = \mathrm{Dom}t$. Then Fm^Λ, the set of strict (or restricted) formulas of the language Λ is the smallest set F such that

(i) $\{R(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{tR-1}) : R \in \underline{R}\} \cup \{\mathbf{v}_i = \mathbf{v}_j : i,j \in \alpha\} \cup \{\mathbf{T}, \mathbf{F}\} \subseteq F$

(ii) $\{\exists \mathbf{v}_i\varphi, \neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi)\} \subseteq F$ if $i \in \alpha$ and $\varphi, \psi \in F$.

Fm_w^Λ, the set of weak (or redundant) formulas is the smallest set satisfying

(iii) $\{R(\mathbf{v}_{j_0}, \mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_{tR-1}}) : R \in \underline{R}, j \in {}^{t(R)}\alpha\} \subseteq F$

besides (i), (ii) above. By formula we mean restricted formula unless explicitly stated otherwise. Truth of a formula (or set of formulas) in a (class of) models is understood as usual. In particular,

$\underline{M} \models \varphi \stackrel{\text{def}}{\iff} (\forall k \in {}^\alpha M)\underline{M} \models \varphi[k]$ and

$\Sigma \models \varphi \stackrel{\text{def}}{\iff} (\forall \underline{M})[\underline{M} \models \Sigma \Rightarrow \underline{M} \models \varphi]$.

$\mathrm{Fvar}(\varphi)$ denotes the set of variables occurring free in φ. For $H \subseteq \alpha$ let

$$\mathrm{Fm}^{\Lambda,H} \stackrel{\text{def}}{=} \{\varphi \in \mathrm{Fm}^\Lambda : \mathrm{Fvar}(\varphi) \subseteq \{\mathbf{v}_i : i \in H\}\}.$$

We often write $\mathrm{Fm}_\alpha^\Lambda$ and $\mathrm{Fm}_\alpha^{\Lambda,H}$ instead of Fm^Λ and $\mathrm{Fm}^{\Lambda,H}$. If $\varphi \in \mathrm{Fm}^{\Lambda,H}$ and $k \in {}^H M$ then

$$\underline{M} \models \varphi[k] \stackrel{\text{def}}{\iff} (\exists g \in {}^\alpha M)[k \subseteq g \quad \text{and} \quad \underline{M} \models \varphi[g]].$$

If $H \subseteq \alpha$ and $\varphi \in \mathrm{Fm}^{\Lambda,H}$ then

$$\tilde{\varphi}^{(H,\underline{M})} \stackrel{\text{def}}{=} \{k \in {}^H M : \underline{M} \models \varphi[k]\}.$$

REMARK 2.1 If $\Lambda = \langle \alpha, t\rangle$ and $\alpha \notin \mathrm{Rng}t$ then for all $\varphi \in \mathrm{Fm}_w^\Lambda$ there is a $\psi \in \mathrm{Fm}^\Lambda$ with $\models \varphi \leftrightarrow \psi$. In this paper this condition usually holds — in those rare cases where it does not, we shall investigate weak formulas separately (for example in the proof of Corollary 4.1). ■

2.4 Proof systems

Let $\Lambda = \langle \alpha, t\rangle$ be a first order language. The inference system $\vdash_\Lambda$ (or simply $\vdash_\alpha$) is the one defined in the Introduction except that: in place of ((4a))...((4d)) we use the (equivalent)

((4)) $\varphi \rightarrow \forall \mathbf{v}_i\varphi$ if $\mathbf{v}_i \notin \mathrm{Fvar}(\varphi)$,

and formulas φ, ψ are in Fm^Λ and $i, j, k \in \alpha$.

If $Ax \subseteq \mathrm{Fm}^\Lambda$ then $\equiv_{Ax}^\Lambda \stackrel{\text{def}}{=} \equiv_{Ax} \stackrel{\text{def}}{=} \{(\varphi, \psi) \in {}^2\mathrm{Fm}^\Lambda : Ax \vdash_\Lambda \varphi \leftrightarrow \psi\}$ and $\equiv^\Lambda \stackrel{\text{def}}{=} \equiv \stackrel{\text{def}}{=} \equiv_0$.

We note that the proof system $\vdash_\Lambda$ is complete in case $\alpha \geq \omega$ or $\mathrm{Rng}t \subseteq 2$.

2.5 Cylindric algebras

Let α be a set. The algebraic type cyl_α has constant symbols $0, 1, \mathrm{d}_{ij}$ $(i, j \in \alpha)$, unary function symbols $-, \mathrm{c}_i$ $(i \in \alpha)$ and binary function symbols $+, \cdot$. CTA_α denotes the class of algebras of type cyl_α.

Let $\underline{A} \in CTA_\alpha$, $K \subseteq CTA_\alpha$ and $\beta \subseteq \alpha$. Then

$\underline{Rd_\beta A} \stackrel{\text{def}}{=} \langle A, +^{\underline{A}}, \cdot^{\underline{A}}, -^{\underline{A}}, 0^{\underline{A}}, 1^{\underline{A}}, \mathrm{c}_i^{\underline{A}}, \mathrm{d}_{ij}^{\underline{A}} \rangle_{i,j\in\beta}$

$\mathbf{Rd}_\beta K \stackrel{\text{def}}{=} \{\underline{Rd_\beta A} : \underline{A} \in K\}$ and $\underline{BlA} \stackrel{\text{def}}{=} \underline{Rd_0 A}$ (the Boolean reduct of $\underline{A}$)

$\Delta^{\underline{A}}(a) \stackrel{\text{def}}{=} \{i \in \alpha : \mathrm{c}_i^{\underline{A}}(a) \neq a\}$ if $a \in A$.

Let X be a set, $\delta : X \longrightarrow \mathrm{Sb}\alpha$ and $K \subseteq CTA_\alpha$. Then

$\mathrm{Cr}_X^{(\delta)} K \stackrel{\text{def}}{=} \bigcap\{R : \underline{F} \stackrel{\text{def}}{=} \underline{\mathrm{Tm}}_X(\mathrm{cyl}_\alpha)/R \in \mathbf{IS}K$ and $(\forall x \in X)\Delta^{\underline{F}}(x) \subseteq \delta x\}$

$\underline{\mathcal{F}}_X^{(\delta)} K \stackrel{\text{def}}{=} \underline{\mathrm{Tm}}_X(\mathrm{cyl}_\alpha)/\mathrm{Cr}_X^{(\delta)} K.$

Derived operations in CTA_α's:

The usual Boolean ones, and

$\mathrm{s}_j^i \tau \stackrel{\text{def}}{=} \mathrm{c}_i(\mathrm{d}_{ij} \cdot \tau).$

The variety $CA_\alpha \subseteq CTA_\alpha$ of α-dimensional cylindric algebras is defined by the following equations (cf. [8] 1.1.1):

C_0 equations defining Boolean algebras
C_1 $\mathrm{c}_i 0 = 0$
C_2 $x \leq \mathrm{c}_i x$
C_3 $\mathrm{c}_i(x \cdot \mathrm{c}_i y) = \mathrm{c}_i x \cdot \mathrm{c}_i y$
C_4 $\mathrm{c}_i \mathrm{c}_j x = \mathrm{c}_j \mathrm{c}_i x$
C_5 $\mathrm{d}_{ii} = 1$
C_6 $\mathrm{d}_{ij} = \mathrm{c}_k(\mathrm{d}_{ik} \cdot \mathrm{d}_{kj})$ if $k \notin \{i, j\}$
C_7 $\mathrm{c}_i(\mathrm{d}_{ij} \cdot x) \cdot \mathrm{c}_i(\mathrm{d}_{ij} \cdot -x) = 0$ if $i \neq j$.

More precisely, $C_1^\alpha \stackrel{\text{def}}{=} \{\mathrm{c}_i 0 = 0 : i \in \alpha\}$, etc. $BA \stackrel{\text{def}}{=} CA_0$ is the class of Boolean algebras.

2.6 Special cylindric algebras

Algebras of formulas. Let $\Lambda = \langle \alpha, t \rangle$ be a language, $\underline{R} \stackrel{\text{def}}{=} \mathrm{Dom}t$. Then $\underline{Fm}^\Lambda \in CTA_\alpha$ is defined (see [8] 4.3) as

$$\underline{Fm}^\Lambda \stackrel{\text{def}}{=} \langle \mathrm{Fm}^\Lambda, \vee, \wedge, \neg, \mathbf{F}, \mathbf{T}, \exists \mathbf{v}_i, \mathbf{v}_i = \mathbf{v}_j \rangle_{i,j\in\alpha},$$

where $\vee : {}^2\mathrm{Fm}^\Lambda \longrightarrow \mathrm{Fm}^\Lambda$ is defined by $\vee(\varphi, \psi) \stackrel{\text{def}}{=} \varphi \vee \psi$ for all $\varphi, \psi \in \mathrm{Fm}^\Lambda$, and similarly for the other operations. Sometimes we write $\underline{Fm}_\alpha^\Lambda$ instead of $\underline{Fm}^\Lambda$. It is easy to see that

$$\underline{Fm}^\Lambda \cong \underline{\mathrm{Tm}}_{\underline{R}}(\mathrm{cyl}_\alpha) \cong \underline{\mathcal{F}}_{\underline{R}} CTA_\alpha.$$

The first of these isomorphisms is defined by

$\tau\mu(R(\mathbf{v}_0,\ldots,\mathbf{v}_{tR-1})) \stackrel{\text{def}}{=} R$ if $R \in \underline{R}$

$\tau\mu(\mathbf{v}_i = \mathbf{v}_j) \stackrel{\text{def}}{=} \mathrm{d}_{ij}$, $\tau\mu(\mathbf{F}) \stackrel{\text{def}}{=} 0$, $\tau\mu(\mathbf{T}) \stackrel{\text{def}}{=} 1$

$\tau\mu(\varphi \vee \psi) \stackrel{\text{def}}{=} \tau\mu(\varphi) \vee \tau\mu(\psi)$, $\tau\mu(\varphi \wedge \psi) \stackrel{\text{def}}{=} \tau\mu(\varphi) \wedge \tau\mu(\psi)$, $\tau\mu(\neg\varphi) \stackrel{\text{def}}{=} -\tau\mu(\varphi)$

$\tau\mu(\exists \mathbf{v}_i \varphi) \stackrel{\text{def}}{=} \mathrm{c}_i \tau\mu(\varphi)$.

Then $\tau\mu : \underline{Fm}^\Lambda \rightarrowtail\!\!\twoheadrightarrow \underline{\mathrm{Tm}_{\underline{R}}}(\mathrm{cyl}_\alpha)$. It is not hard to check that $\underline{Fm}^\Lambda / \equiv_{Ax} \in CA_\alpha$ if $Ax \subseteq \mathrm{Fm}^\Lambda$. Moreover (see [8] 4.3.25) $\underline{Fm}^\Lambda / \equiv \; \cong \underline{\mathcal{F}}^{(t)}_{\underline{R}} CA_\alpha$. This means that there is a close connection between the class CA_α and the proof system $\vdash_\alpha$: the collection of CA_α-axioms is an algebraic version of the proof system $\vdash_\alpha$. One corollary (see [8] 4.3.28(i)) is that

$$CA_\alpha = \mathbf{I}\{\underline{Fm}^\Lambda / \equiv_{Ax} : \Lambda = \langle \alpha, t\rangle \quad \text{is a language,} \quad Ax \subseteq \mathrm{Fm}^\Lambda\}$$

for $\alpha < \omega$. This is a kind of representation theorem for CA_α.

Atom-structures, complex algebras. For α an arbitrary set, cat_α is the relational type with binary relations T_i and unary relations E_{ij} for all $i, j \in \alpha$. Let $\underline{B} = \langle B, T_i^{\underline{B}}, E_{ij}^{\underline{B}}\rangle_{i,j\in\alpha}$ be a model of type cat_α. Then $\underline{CmB} \in CTA_\alpha$, the complex algebra of $\underline{B}$ is

$$\underline{CmB} \stackrel{\text{def}}{=} \langle \mathrm{Sb}B, \cup, \cap, \setminus, 0, B, T_i^{\underline{B}\star}, E_{ij}^{\underline{B}}\rangle_{i,j\in\alpha}.$$

If $K \subseteq \mathrm{Mod}(\mathrm{cat}_\alpha)$ then

$$\mathbf{Cm}K \stackrel{\text{def}}{=} \{\underline{CmA} : \underline{A} \in K\}.$$

Let $\mathrm{At}_\alpha \subseteq \mathrm{Mod}(\mathrm{cat}_\alpha)$ be the class of those models $\underline{B} = \langle B, T_i, E_{ij}\rangle_{i,j\in\alpha}$ which satisfy (i)...(v) below for all $i, j, k \in \alpha$:

(i) T_i is an equivalence relation on B

(ii) $T_i \mid T_j = T_j \mid T_i$

(iii) $E_{ii} = B$

(iv) $E_{ij} = T_k^\star(E_{ik} \cap E_{kj})$ if $k \notin \{i,j\}$

(v) $T_i \cap {}^2E_{ij} \subseteq \mathrm{Id}$ if $i \neq j$.

We note that (iv) can be replaced with

(iv') $E_{ij} = E_{ji}$, $E_{ik} \cap E_{kj} \subseteq E_{ij}$, $E_{ij} = T_k^\star E_{ij}$ and $E_{ij} \subseteq T_k^\star(E_{ik} \cap E_{kj})$ if $k \notin \{i,j\}$.

Elements of At_α are called (cylindric) atomstructures. In [8] 2.7.43(ii), 2.7.40 it is shown that

$$CA_\alpha = \mathbf{ISCm}\mathrm{At}_\alpha.$$

This is again a kind of representation theorem for CA_α's (which, in the $\alpha = 0$ case, coincides with the representation theorem of BA's).

Cylindric set algebras. Let α be an arbitrary set and let $V \subseteq {}^\alpha U$ for some set U. For $i, j \in \alpha$ let

$$\mathrm{D}_{ij}^{[V]} \stackrel{\text{def}}{=} \{s \in V : s(i) = s(j)\}$$

$$T_i^{[V]} \stackrel{\text{def}}{=} \{(s,z) \in {}^2V : s \restriction (\alpha \setminus \{i\}) = z \restriction (\alpha \setminus \{i\})\}$$

$$\mathrm{C}_i^{[V]}X \stackrel{\text{def}}{=} T_i^{[V]\star}X = \{s(i/u) \in V : s \in X, u \in U\}.$$

$$\underline{At}(V) \stackrel{\text{def}}{=} \langle V, T_i^{[V]}, \mathrm{D}_{ij}^{[V]}\rangle_{i,j\in\alpha}$$

$$\underline{Sb}V \stackrel{\text{def}}{=} \underline{CmAt}(V) = \langle \mathrm{Sb}V, \cup, \cap, \setminus, V, \mathrm{C}_i^{[V]}, \mathrm{D}_{ij}^{[V]}\rangle_{i,j\in\alpha}$$

$$Crs_\alpha \stackrel{\text{def}}{=} \mathbf{S}\{\underline{Sb}V : (\exists U) V \subseteq {}^\alpha U\}.$$

Members of Crs_α are called cylindric relativized set algebras. For $V \subseteq {}^\alpha U$ we let

$$\mathrm{base}(V) \stackrel{\text{def}}{=} \bigcup\{\mathrm{Rng}s : s \in V\}$$

and if $\underline{A} \in Crs_\alpha$ then

$$\mathrm{base}(\underline{A}) \stackrel{\text{def}}{=} \mathrm{base}(1^{\underline{A}}).$$

The class of cylindric set algebras is defined by

$$Cs_\alpha \stackrel{\text{def}}{=} \mathbf{S}\{\underline{Sb}^\alpha U : U \quad \text{is a set}\}$$

and the class of representable cylindric algebras is

$$RCA_\alpha \stackrel{\text{def}}{=} \mathbf{HSP}Cs_\alpha.$$

We note that if $\alpha \geq 2$ then

$$RCA_\alpha = \mathbf{I}Gs_\alpha = \mathbf{SP}Cs_\alpha$$

where

$$\begin{aligned} Gs_\alpha \stackrel{\text{def}}{=} \; & \mathbf{S}\{\underline{Sb}(\bigcup\{{}^\alpha U_i : i \in I\}) : I \quad \text{is a set, and} \\ & (U_i : i \in I) \quad \text{is system of pairwise disjoint sets}\}. \end{aligned}$$

2.7 Recursion-theoretic notions

In this paper the notion of decidability will be extended from subsets of ω and sets of terms to other sets (e.g. to sets of finite algebras of finite types) in the intuitively natural way. Similarly for recursive functions. The expressions "decidable", "recursive" and "computable" will be used interchangeably. Related terms will sometimes be used inaccurately within proofs, thus we write "$N(\tau)$ is computable" instead of "the function N is computable", or "it is decidable whether there exists a τ-tree" instead of "there is a recursive function which decides whether there exists a τ-tree, for each τ".

Let K be a class of structures. (Thus K may be a class of algebras.)

(i) K is said to be strongly decidable if its similarity type is finite and there is a recursive function $f : \omega \longrightarrow \omega$ such that

$$\text{(a)} \qquad (\forall \underline{A} \in K)(\forall X \subseteq_\omega A)(\exists \underline{B} \in K)[\underline{A} \restriction X = \underline{B} \restriction X \text{ and } |B| \leq f(|X|)]$$

and

$$\text{(b)} \qquad \{\underline{A} \in K : A \in \omega\} \quad \text{is decidable.}$$

(ii) $\mathbf{F}K \stackrel{\text{def}}{=} \{\underline{A} \in K : |A| < \omega\}$.

(iii) The set of equations valid in K is denoted by EqK and the set of quasi-equations valid in K is denoted by $QeqK$.

REMARK 2.2 Let K be a class of algebras.
(i) If K is strongly decidable, then it is easy to see that not only EqK, but the set of universal formulas valid in K is decidable. In particular, $QeqK$ is decidable and thus the word-problem of K is solvable. Besides, EqK = Eq$\mathbf{F}K$, moreover K and $\mathbf{F}K$ cannot be distinguished even by universal formulas.
(ii) Connections between EqK = Eq$\mathbf{F}K$ and the decidablity of EqK: It was observed in Taylor [23], p.26 that if K is finitely based (or more generally: recursively based) and EqK = Eq$\mathbf{F}K$, then EqK is decidable. (Indeed, it is not hard to see: EqK is enumerable since K is finitely based, and the equations that are not valid in K are also enumerable since EqK = Eq$\mathbf{F}K$ and K is finitely based.) We cannot weaken the condition "K is recursively based" to "EqK is enumerable": Let $N \subseteq \omega$ be enumerable but undecidable and let $E \stackrel{\text{def}}{=} \{gf^n0 = f^n0 : n \in N\}$ (thus f and g are unary function-symbols and 0 is a constant-symbol) and let K be the variety defined by E. Then it is easy to see that EqK is enumerable but undecidable and EqK = Eq$\mathbf{F}K$. This example also shows that condition (b) cannot be removed from (i) of the definition of strong decidability (without affecting the truth of the implication "K is strongly decidable $\Rightarrow$ EqK is decidable"). Conversely, it is quite easy to construct a class of algebras K such that EqK is finitely based and decidable but Eq$K \neq$ Eq$\mathbf{F}K$. For example, let $E = \{gfx = x, g0 = 0\}$ and let K be the variety defined by E. Then it is not hard to see that EqK is decidable, $\mathbf{F}K \models 0 = f0$ but $K \not\models 0 = f0$.
(iii) Henkin proved the strong decidability of CA_2, and even that of RCA_2, see [8] 2.5.4, 4.2.8. ∎

3 The decidability of the equational theory of non-commutative cylindric algebras

NCA_α, the class of "non-commutative CA_α's" was first defined and investigated by R. J. Thompson.

DEFINITION 3.1 Let α be an arbitrary set. NCA_α is the class of those $\underline{A} \in CTA_\alpha$ which satisfy the identities defining CA_α with the exception of C_4. That is,

$$NCA_\alpha \stackrel{\text{def}}{=} \{\underline{A} \in CTA_\alpha : \underline{A} \models C_0 \cup C_1^\alpha \cup C_2^\alpha \cup C_3^\alpha \cup C_5^\alpha \cup C_6^\alpha \cup C_7^\alpha\}.$$

■

Below, we shall prove the decidability of the equational theory of NCA_α and that there is no equation valid in all finite NCA_α but not in all NCA_α, if $\alpha < \omega$. Nevertheless, NCA_α is not strongly decidable, since there is a quasi-equation valid in all finite NCA_α but not in all NCA_α.

THEOREM 3.1 *Let α be an arbitrary set.*

(i) EqNCA_α *is decidable (provided α is decidable).*

(ii) Eq$NCA_\alpha =$ EqFNCA_α, *if $\alpha < \omega$.*

(iii) *Qeq$NCA_\alpha \neq$ Qeq*F*NCA_α, if $|\alpha| \geq 3$. Thus NCA_α is not strongly decidable if $\alpha \geq 3$.*

Proof: First we introduce some notation. We will have to work a lot with terms in the constructions below. Thus we find it convenient to introduce the following conventions. Let X be a set and fix $x \in X$. We write Tm instead of $\mathrm{Tm}_X(cil_\alpha)$. The elements of Tm should be thought of as being built up from X by the operations $-$, $\cdot$, c_i, d_{ij} $(i, j < \alpha)$ (thus 0, 1 and + are regarded as derived operations).

Define the algebraic type t_α as follows:

$$t_\alpha \stackrel{\text{def}}{=} \{(\mathrm{t}_{in}, 1) : i \in \alpha, n \in \alpha \cup \mathrm{Tm}\},$$

so that all function-symbols in t_α are unary. We will simply write $\mathrm{Tm}(t_\alpha)$ in place of $\mathrm{Tm}_{\{x\}}(t_\alpha)$. If Γ is a set of terms, Subterm(Γ) denotes the set of subterms of the elements of Γ; we write Subterm(σ) instead of Subterm($\{\sigma\}$). $\mathrm{Subterm}(\sigma)' \stackrel{\text{def}}{=} \{\delta, -\delta : \delta \in \mathrm{Subterm}(\sigma)\}$.

If $\sigma \in \mathrm{Tm} \cup \mathrm{Tm}_X(t_\alpha)$, the *length* of σ is denoted by $||\sigma||$ and is defined by the following clauses:

$||y|| \stackrel{\text{def}}{=} 1$, if $y \in X$

$$||\mathrm{t}_{in}w|| \stackrel{\mathrm{def}}{=} ||w|| + 1, \text{ if } \mathrm{t}_{in}w \in \mathrm{Tm}_X(t_\alpha)$$

$$||\mathrm{d}_{ij}|| \stackrel{\mathrm{def}}{=} 1$$

$$||-\sigma|| \stackrel{\mathrm{def}}{=} ||\mathrm{c}_i\sigma|| \stackrel{\mathrm{def}}{=} ||\sigma|| + 1$$

$$||\delta \cdot \sigma|| \stackrel{\mathrm{def}}{=} ||\delta|| + ||\sigma||, \text{ if } \sigma,\ \delta \in \mathrm{Tm} \text{ and } i,\ j \in \alpha.$$

$\mathrm{Eqrel}(H)$ denotes the set of equivalence relations of H. If $e \in \mathrm{Eqrel}(H)$ and $G \subseteq H$, then $e \restriction G \stackrel{\mathrm{def}}{=} {}^2G \cap e$. We note that if $e \in \mathrm{Eqrel}(H)$ and $h \in H$, then $e^\star\{h\}$ is the e-equivalence class of h. Let $e \in \mathrm{Eqrel}(H)$ and $g \in H$. Then

$$e(g/h) \stackrel{\mathrm{def}}{=} [e \restriction (H \setminus \{g\})] \cup {}^2(\{g\} \cup e^\star\{h\}),$$

that is, $e(g/h)$ is the equivalence relation on H obtained from e by moving g into the equivalence class of h if $h \in H$ and putting g into a separate equivalence class otherwise. (We note that $e^\star\{h\} = 0$ if $h \notin H$.) i is said to be *singular* in e if $e^\star\{i\} = \{i\}$. In the sequel α will be assumed to consist of ordinals. Then α is well-ordered by $\subseteq$, and for $\Gamma \subseteq \alpha$, $\min(\Gamma)$ denotes the minimal element of Γ with respect to this ordering.

DEFINITION 3.2 Let $e \in \mathrm{Eqrel}(\alpha)$ and $\tau \in \mathrm{Tm}$.
(i) $\mathrm{E}^e : \mathrm{Tm}(t_\alpha) \longrightarrow \mathrm{Eqrel}(\alpha)$ is defined by the following recursion:

$$\mathrm{E}^e(x) \stackrel{\mathrm{def}}{=} e, \text{ and}$$

$$\mathrm{E}^e(\mathrm{t}_{in}w) \stackrel{\mathrm{def}}{=} \mathrm{E}^e(w)(i/n) \text{ if } \mathrm{t}_{in}w \in \mathrm{Tm}(t_\alpha).$$

(ii) Let $i \in \alpha$, $n \in \alpha \cup \mathrm{Tm}$ and $w \in \mathrm{Tm}(t_\alpha)$. w is said to start with i if $w = \mathrm{t}_{il}w'$ for some l and w'.

$\mathrm{t}^e_{in} : \mathrm{Tm}(t_\alpha) \longrightarrow \mathrm{Tm}(t_\alpha)$ is defined as follows: Let

$$k \stackrel{\mathrm{def}}{=} \begin{cases} \min(\mathrm{E}^e(w)^\star\{n\}), & \text{if } n \in \alpha, \\ n, & \text{if } n \notin \alpha. \end{cases}$$

Then

$$\mathrm{t}^e_{in}w \stackrel{\mathrm{def}}{=} \begin{cases} w, & \text{if } (i,n) \in \mathrm{E}^e(w); \text{ otherwise} \\ \mathrm{t}_{ik}w, & \text{if } w \text{ does not start with } i \\ \mathrm{t}_{ik}w, & \text{if } (\exists l)w = \mathrm{t}_{il}w' \text{ and } (i,k) \notin \mathrm{E}^e(w') \\ w', & \text{if } (\exists l)w = \mathrm{t}_{il}w' \text{ and } (i,k) \in \mathrm{E}^e(w'). \end{cases}$$

(iii) P is said to be a τ-tree if

1. $P \subseteq \mathrm{Tm}(t_\alpha) \times \mathrm{Tm}$, and if $(w,\sigma) \in P$, then $||w|| \leq ||\tau||$, $(\forall i,n)[\mathrm{t}_{in}$ occurs in $w \Rightarrow n \in \mathrm{Subterm}(\tau)' \cup \alpha]$, and $\sigma \in \mathrm{Subterm}(\tau)' \cup \{\mathrm{d}_{ij}, -\mathrm{d}_{ij} : i,j \in \alpha\}$.

2.

$$
\begin{array}{ll}
(a) & (x,\tau) \in P \\
(b) & e \stackrel{\text{def}}{=} \{(i,j) : (x, \mathrm{d}_{ij}) \in P\} \in \mathrm{Eqrel}(\alpha) \\
(c) & \text{for all } w \in \mathrm{Tm}(t_\alpha),\ i,j \in \alpha \text{ and } \sigma,\delta \in \mathrm{Tm}, \text{ we have} \\
(c1) & (w,\sigma) \in P \Rightarrow \{(w,\mathrm{d}_{ij}) : (i,j) \in \mathrm{E}^e(w)\} \cup \{(w, -\mathrm{d}_{ij}) : \\
 & \qquad (i,j) \notin \mathrm{E}^e(w)\} \subseteq P \\
(c2) & (w, \sigma \cdot \delta) \in P \Rightarrow \{(w,\sigma),(w,\delta)\} \subseteq P \\
 & (w, -(\sigma\cdot\delta)) \in P \Rightarrow [(w,-\sigma) \in P \text{ or } (w,-\delta) \in P] \\
 & (w, -(-\sigma)) \in P \Rightarrow (w,\sigma) \in P \\
 & (w, -\mathrm{c}_i\sigma) \in P \Rightarrow \{(\mathrm{t}^e_{in} w, -\sigma) : n \in \alpha \text{ or } \mathrm{t}^e_{in} w \in \mathrm{Dom}P\} \subseteq P \\
 & (w, \mathrm{c}_i\sigma) \in P \Rightarrow (\exists n \in \alpha \cup \mathrm{Subterm}(\tau)')(\mathrm{t}^e_{in} w, \sigma) \in P \\
(c3) & (w,\sigma) \in P \Rightarrow (w,-\sigma) \notin P
\end{array}
$$

■

The intuitive meaning of τ-trees will be explained in Remark 3.4(ii) below.

PROPOSITION 3.1 *Let* $\tau \in \mathrm{Tm}$.

(i) $NCA_\alpha \not\models \tau = 0 \Longleftrightarrow$ *there is a* τ*-tree.*

(ii) *There is a recursive function* $N : \mathrm{Tm} \longrightarrow \omega$ *such that* $[NCA_\alpha \models \tau = 0 \Longleftrightarrow \{\underline{A} \in NCA_\alpha : |A| \leq N(\tau)\} \models \tau = 0]$, *if* $\alpha < \omega$.

Proof: To prove the proposition we need the definition of atomstructures of NCA_α's first.

DEFINITION 3.3 Let $\underline{B} = \langle B, T_i, E_{ij}\rangle_{i,j\in\alpha}$ be such that $T_i \subseteq {}^2B$ and $E_{ij} \subseteq B$ for all $i,j \in \alpha$.

(a) $\underline{B}$ is said to be a partial N-atomstructure, $\underline{B} \in pNAt_\alpha$, if $\underline{B}$ satisfies conditions (i)...(iv) below for all $i,j,k \in \alpha$.

(i) $T_i \in \mathrm{Eqrel}(B)$

(ii) $E_{ii} = B$, $E_{ij} = E_{ji}$, $E_{ik} \cap E_{kj} \subseteq E_{ij}$

(iii) $E_{ij} = T_k^\star E_{ij}$ if $k \notin \{i,j\}$

(iv) $T_i \cap {}^2E_{ij} \subseteq \mathrm{Id}$ if $i \neq j$.

(b) $\underline{B}$ is said to be an N-atomstructure, $\underline{B} \in NAt_\alpha$, if besides (i)...(iv) above, $\underline{B}$ satisfies condition (v) below for all $i,j,k \in \alpha$.

(v) $E_{ij} \subseteq T_k^\star(E_{ik} \cap E_{kj})$.

(c) Let $\underline{B} \in pNAt_\alpha$. Then $\underline{B} = \langle B, T_i^{\underline{B}}, E_{ij}^{\underline{B}} \rangle_{i,j \in \alpha}$, that is, its universe is denoted by B and its relations are distinguished by the superscript $\underline{B}$, unless explicitly stated otherwise. The conventions governing the use of other underlined letters are similar.

(d) Let $\underline{B} \in pNAt_\alpha$, $b \in B$ and $i,j \in \alpha$. Then

$$E^{\underline{B}}(b) \stackrel{\text{def}}{=} \{(i,j) \in {}^2\alpha : b \in E_{ij}^{\underline{B}}\} \text{ and}$$

$$t_{ij}^{\underline{B}} \stackrel{\text{def}}{=} \{(a,b) \in {}^2B : aT_i^{\underline{B}}b \text{ and } b \in E_{ij}^{\underline{B}}\} \text{ if } i \neq j;\ t_{ii}^{\underline{B}} \stackrel{\text{def}}{=} \mathrm{Id}_B.$$

■

PROPOSITION 3.2 $NCA_\alpha = \mathbf{ISCm}NAt_\alpha$.

Proof: The statement follows from [8] 2.7.5, 2.7.14 and 2.7.34. The proof is patterned after that of [8] 2.7.43 (ii) and is routine. ■

REMARK 3.1 The following facts are easy to check so we omit the proofs. Item (3) will not be used in the sequel, it is included here only in order to help the reader's intuition grasp why things are defined the way they are.

(1) NAt_α is an extension of the class At_α defined in section 2, that is, we have omitted from the definition of At_α the condition "$T_i \mid T_j = T_j \mid T_i$ for all $i,j \in \alpha$". Furthermore, we have $pNAt_\alpha = \{\underline{A} \upharpoonright X : \underline{A} \in NAt_\alpha, X \subseteq A\}$ (here "$\supseteq$" is easily verified and "$\subseteq$" follows e.g. from Lemma 3.3 below). Another example for a $pNAt_\alpha$ is the structure $\underline{At}(V)$ defined in section 2: If $V \subseteq {}^\alpha U$, then it is easy to check that $\underline{At}(V) \in pNAt_\alpha$ and $[\underline{At}(V) \in NAt_\alpha \iff (\forall s \in V)(\forall i,j \in \alpha)s(i/s_j) \in V]$. Let $\underline{A} \stackrel{\text{def}}{=} \underline{At}(V)$, $s \in V$ and $i,j \in \alpha$. Then it is easy to verify that $E^{\underline{A}}(s) = \ker s$ and $t_{ij}^{\underline{A}}(s) = s(i/s_j)$. We note however, that the class $\{\underline{At}(V) : (\exists U)V \subseteq {}^\alpha U\} \subseteq pNAt_\alpha$ satisfies much – in fact, infinitely – more regularity than $pNAt_\alpha$, c.f. Theorem 4.1(iii) below.

(2) Let $\underline{B} \in pNAt_\alpha$. Then it is not hard to check that for all $i,j \in \alpha$ and a, $b \in B$ we have

$E^{\underline{B}} : B \longrightarrow \mathrm{Eqrel}(\alpha)$,

$t_{ij}^{\underline{B}}$ is a unary partial function from B to B, and

$[(\forall i,j \in \alpha)t_{ij}^{\underline{B}}$ is total (that is, $\mathrm{Dom}t_{ij}^{\underline{B}} = B)] \iff \underline{B} \in NAt_\alpha$,

$E^{\underline{B}}(t_{ij}^{\underline{B}}b) = E^{\underline{B}}(b)(i/j)$; and

if $aT_i^{\underline{B}}b$, $a \neq b$, then $[E^{\underline{B}}(b) \upharpoonright (\alpha \setminus \{i\}) = E^{\underline{B}}(a) \upharpoonright (\alpha \setminus \{i\})$ and $E^{\underline{B}}(b) = E^{\underline{B}}(a) \Rightarrow i$ is singular in $E^{\underline{B}}(b)]$.

(3) Let $t'_\alpha \stackrel{\text{def}}{=} \{(t_{ij}, 1) : i,j \in \alpha\}$ be an algebraic type. Define a quasi-variety of type t'_α by the following formulas:

For all $i,j,k,l \in \alpha$

(A) $t_{ij}t_{ik}x = t_{ij}x$, if $i \neq j$

(B) $t_{ii}x = x$, $t_{ij}t_{ji}x = t_{ji}x$, $t_{ik}t_{ij}t_{kj}x = t_{ij}t_{kj}x$

(C) $t_{kl}t_{ij}x = t_{ij}x \leftrightarrow t_{kl}x = x$, if $i \notin \{k,l\}$.

Let $\underline{A} = \langle A, t_{ij}\rangle_{i,j\in\alpha} \in V_\alpha$ and $E^{\underline{A}}(a) \stackrel{\text{def}}{=} \{(i,j) \in {}^2\alpha : t_{ij}a = a\}$ if $a \in A$. Then identities (B) and (A),(C) express $E^{\underline{A}}(a) \in \text{Eqrel}(\alpha)$ and $E^{\underline{A}}(t_{ij}a) = E^{\underline{A}}(a)(i/j)$, respectively. Let $\underline{Rd(t'_\alpha)B} \stackrel{\text{def}}{=} \langle B, t_{ij}^{\underline{B}}\rangle_{i,j\in\alpha}$ if $\underline{B} \in NAt_\alpha$. Then it can be shown that $V_\alpha = \{\underline{Rd(t'_\alpha)B} : \underline{B} \in NAt_\alpha\}$ and $V_\alpha \neq \mathbf{H}V_\alpha$, thus V_α is not a variety. Moreover, NAt_α and V_α are definitionally equivalent for $\alpha \geq 2$ in the sense of [8], Part I p.56.: The first order (quantifier-free) definitions are as follows.

Let i, $j \in \alpha$.

$$t_{ij}x = y \Leftrightarrow (xT_iy \wedge E_{ij}(y)) \text{ if } i \neq j, \text{ and } t_{ii}x = y \Leftrightarrow x = y,$$

$$xT_iy \Leftrightarrow t_{ij}x = t_{ij}y \text{ (where } j \in \alpha \setminus \{i\} \text{ is fixed)}, \; E_{ij}(x) \Leftrightarrow t_{ij}x = x.$$

Let $\alpha < \omega$ (for simplicity) throughout the remaining part of this remark. By Lemma 3.3 (to be proved later) both NAt_α and V_α are strongly decidable, thus for example, the word-problem for V_α is solvable. But $\mathbf{Cm}NAt_\alpha$ is not strongly decidable by Theorem 3.1(iii) (and the proof shows that $\mathbf{Cm}V_\alpha$ is not strongly decidable either). Indeed, there is a simple decision procedure based on a kind of normal form for $\text{Eq}V_\alpha$ that we shall now describe.

On the definition of the function $t_{in}^e : \text{Tm}(t_\alpha) \longrightarrow \text{Tm}(t_\alpha)$: One easily checks that if $\underline{A} \in V_\alpha$, $a \in A$, $E^{\underline{A}}(a) = e$ and $t_{in}w \in \text{Tm}(t'_\alpha)$, then $\underline{A} \models t_{in}^e w = t_{in}w[a]$. Moreover, t_{in}^e was defined so that the equations valid in V_α could be described with the help of the notion of normal-form yielded by it: Let $e \in \text{Eqrel}(\alpha)$, and define the function $n^e : \text{Tm}_X(t'_\alpha) \longrightarrow \text{Tm}_X(t'_\alpha)$ as follows:

$$n^e(y) \stackrel{\text{def}}{=} y \text{ if } y \in X, \text{ and}$$

$$n^e(t_{ij}w) \stackrel{\text{def}}{=} t_{ij}^e n^e w \text{ if } i,\, j \in \alpha \text{ and } w \in \text{Tm}_X(t'_\alpha).$$

Then n^e is a computable (i.e. recursive) function and it can be shown that for all w, $z \in \text{Tm}_X(t'_\alpha)$

$$(***) \qquad V_\alpha \models w = z \Leftrightarrow (\forall e \in \text{Eqrel}(\alpha))n^e(w) = n^e(z).$$

This gives a simple decision-procedure for the equations valid in V_α. The proof of $(***)$ can be reconstructed from certain parts of the proof of Theorem 3.1 below. It would be interesting to know whether the decidability of $\text{Eq}NCA_\alpha$ can be derived from that of $\text{Eq}V_\alpha$, that is, if there is a recursive function tr on the class of equations of type cil_α to the equations of type t'_α such that $NCA_\alpha \models q \Leftrightarrow V_\alpha \models tr(q)$, for all equations q. ∎

We return to the proof of Proposition 3.1.

(I) Proof of "$NCA_\alpha \not\models \tau = 0 \Rightarrow$ there is a τ-tree":

Assume $NCA_\alpha \not\models \tau = 0$. Then, by Proposition 3.2 there is an $\underline{A} \in NAt_\alpha$ and $k : X \longrightarrow \mathrm{Sb}A$ such that $\underline{C} \stackrel{\text{def}}{=} \underline{CmA} \not\models \tau = 0[k]$, that is, $\tau^{\underline{C}}[k] \neq 0$. Let $a \in \tau^{\underline{C}}[k]$ and $e \stackrel{\text{def}}{=} E^{\underline{A}}(a)$. We define P_m and h_m by recursion so that they will satisfy condition $(*)$ below: Let $D_m \stackrel{\text{def}}{=} \mathrm{Dom}P_m$ and $R_m \stackrel{\text{def}}{=} \mathrm{Rng}\bar{P}_m \setminus (\{\mathrm{d}_{ij}, -\mathrm{d}_{ij} : i,j \in \alpha\} \cup \{x, -x : x \in X\})$, where $\bar{P}_0 \stackrel{\text{def}}{=} P_0$, and $\bar{P}_m \stackrel{\text{def}}{=} P_m \setminus P_{m-1}$ if $m > 0$.

Let $(*)$ be the conjunction of $*1 \ldots *7$ below:

$*1$ $D_m \subseteq \{w \in \mathrm{Tm}(t_\alpha) : ||w|| \leq m+1\}$; $D_m = \mathrm{Subterm}(D_m)$; $t_{in}w \in D_m \Rightarrow n \in \mathrm{Subterm}(\tau)' \cup \alpha$

$*2$ $||\sigma|| \leq ||\tau|| - m$ if $\sigma \in R_m$; $R_m \subseteq \mathrm{Subterm}(\tau)'$

$*3$ $h_m : D_m \longrightarrow A$

$*4$ $h_m w \in \sigma^{\underline{C}}[k]$ if $(w, \sigma) \in P_m$

$*5$ $E^{\underline{A}}(h_m w) = E^e(w)$ if $w \in D_m$

$*6$ $h_m(t_{i\sigma}w) \in \sigma^{\underline{C}}[k]$ if $t_{i\sigma}w \in D_m$ and $\sigma \in \mathrm{Tm}$

$*7$ $h_m w T_i^{\underline{A}} h_m(t_{in}w)$ if $i \in \alpha$, $n \in \alpha \cup \mathrm{Tm}$ and $t_{in}w \in D_m$.

Let $P_0 \stackrel{\text{def}}{=} \{(x,\tau)\} \cup \{(x, \mathrm{d}_{ij}) : (i,j) \in e\} \cup \{(x, -\mathrm{d}_{ij}) : (i,j) \in {}^2\alpha \setminus e\}$, $h_0 \stackrel{\text{def}}{=} \{(x,a)\}$. Then P_0, h_0 satisfies $(*)$. Suppose that P_m, h_m satisfies $(*)$. Let

$$H_m \stackrel{\text{def}}{=} \{(w,i,\sigma) : (w, \mathrm{c}_i\sigma) \in P_m, (\forall n \in \alpha \cup \mathrm{Tm})[t^e_{in}w \in D_m \Rightarrow h_m(t^e_{in}w) \notin \sigma^{\underline{C}}[k]]\}.$$

Let $b : H_m \longrightarrow A$ be such that for all (w,i,σ), $(w',i,\sigma) \in H_m$

$$(h_m w) T_i^{\underline{A}} b(w,i,\sigma),\ b(w,i,\sigma) \in \sigma^{\underline{C}}[k] \text{ and}$$

$$b(w,i,\sigma) = b(w',i,\sigma) \text{ if } (h_m w) T_i^{\underline{A}} (h_m w').$$

There *is* such a function b. Define $f : H_m \longrightarrow \mathrm{Tm}(t_\alpha)$ by

$$f(w,i,\sigma) \stackrel{\text{def}}{=} \begin{cases} t^e_{i\sigma}w, & \text{if } E^{\underline{A}}(b(w,i,\sigma))^\star\{i\} = \{i\} \\ t^e_{ik}w, & \text{if } k = \min(E^{\underline{A}}(b(w,i,\sigma))^\star\{i\} \setminus \{i\}). \end{cases}$$

Let

$$G_m \stackrel{\text{def}}{=} \{(w,i) : (\exists \sigma \in \mathrm{Tm})(w, -\mathrm{c}_i\sigma) \in P_m\}$$

$$W \stackrel{\text{def}}{=} \{f(w,i,\sigma) : (w,i,\sigma) \in H_m\} \cup \{t^e_{in}w : (w,i) \in G_m, n \in \alpha\}$$

$$\begin{aligned} h_{m+1} \stackrel{\text{def}}{=} & h_m \cup \{(f(w,i,\sigma), b(w,i,\sigma)) : (w,i,\sigma) \in H_m\} \\ & \cup \{(t^e_{in}w, t^{\underline{A}}_{in}h_m w) : (w,i) \in G_m, n \in \alpha\} \end{aligned}$$

$$
\begin{aligned}
P_{m+1} \stackrel{\text{def}}{=} & \{(w, \mathrm{d}_{ij}) : w \in W, (i,j) \in E^e(w)\} \\
& \cup \{(w, -\mathrm{d}_{ij}) : w \in W, (i,j) \in {}^2\alpha \setminus E^e(w)\} \\
& \cup \{(t^e_{in}w, \sigma) : (w, \mathrm{c}_i\sigma) \in P_m, t^e_{in}w \in D_m, h_m(t^e_{in}w) \in \sigma^{\underline{C}}[k]\} \\
& \cup \{(f(w,i,\sigma), \sigma) : (w,i,\sigma) \in H_m\} \\
& \cup \{(t^e_{in}w, -\sigma) : (w, -\mathrm{c}_i\sigma) \in P_m, n \in \alpha \text{ or } t^e_{in}w \in D_m \cup W\} \\
& \cup \{(w,\sigma) : (w, \sigma \cdot \delta) \in P_m\} \cup \{(w,\delta) : (w, \sigma\cdot\delta) \in P_m\} \\
& \cup \{(w, -\sigma) : (w, -(\sigma \cdot \delta)) \in P_m, h_m w \in (-\sigma)^{\underline{C}}[k]\} \\
& \cup \{(w, -\delta) : (w, -(\sigma \cdot \delta)) \in P_m, h_m w \in (-\delta)^{\underline{C}}[k]\} \\
& \cup \{(w,\sigma) : (w, -(-\sigma)) \in P_m\} \cup P_m.
\end{aligned}
$$

Now we show that P_{m+1} and h_{m+1} satisfy $(*)$. It is clear from the definitions that $*1$ and $*2$ are satisfied. Next we show that h_{m+1} is a function. To this end let us first prove propositions (1)...(5) below. (In the proof we simply write h instead of h_m.)

(1) $\quad h(t_{ik}w) = t^{\underline{A}}_{ik}hw$ if $t_{ik}w \in D_m$ and $k \in \alpha$.

Indeed, if $t_{ik}w \in D_m$, then $w \in D_m$ by $*1$ (and thus h is defined on w), and $*7$ and $*5$ gives $h(w)T^{\underline{A}}_i h(t_{ik}w) \in \mathrm{d}^{\underline{A}}_{ik}$, that is, $h(t_{ik}w) = t^{\underline{A}}_{ik}hw$ (since $\underline{A} \in NAt_\alpha$) if $i \neq k$. One shows by induction that $t_{ik}w \notin D_m$ if $i = k$. ■(1)

(2) $\quad t^{\underline{A}}_{in}hw = t^{\underline{A}}_{kl}hw''$ if $t^e_{in}w = t_{kl}w''$, $\quad w \in D_m$ and $i, n \in \alpha$.

Indeed, let $w \in D_m$, i, $n \in \alpha$ and $t^e_{in}w = t_{kl}w''$. If $(i,n) \in E^e(w)$, then $t^e_{in}w = w \in D_m$ and $*5$ gives $t^{\underline{A}}_{in}hw = hw$, so we have $t^{\underline{A}}_{in}hw = hw = h(t_{kl}w'') = t^{\underline{A}}_{kl}hw''$ by (1). Suppose that $(i,n) \notin E^e(w)$, and let $k = \min E^e(w)^\star\{n\}$. Suppose that $t^e_{in}w = t_{ik}w''$, where either $w'' = w$ or $(\exists l \in \alpha \cup \mathrm{Tm})w = t_{il}w''$. In both cases $w'' \in D_m$ (by $*1$ since $w \in D_m$) and $hwT^{\underline{A}}_i hw''$ by $*7$, thus we have $t^{\underline{A}}_{in}hw = t^{\underline{A}}_{ik}hw = t^{\underline{A}}_{ik}hw''$ since $(n,k) \in E^e(w) = E^{\underline{A}}(hw)$. By the definition of t^e_{in} the only case not covered so far is $t^e_{in}w = w'$, where $(\exists l)w = t_{il}w'$ and $(i,k) \in E^e(w')$. Then $w' \in D_m$ and $(i,n) \in E^e(w')$, $hwT^{\underline{A}}_i hw'$, so $t^{\underline{A}}_{in}hw = hw'$. If $w' = t_{kl}w''$, then (1) gives $hw' = t^{\underline{A}}_{kl}hw''$. ■(2)

(3) $\quad h(t^e_{in}w) = t^{\underline{A}}_{in}hw$ if $w, t^e_{in}w \in D_m$.

Indeed, if $t^e_{in}w = t_{kl}w''$ for some k, l, w'', then $w \in D_m$, and (2), (1) gives $t^{\underline{A}}_{in}hw = t^{\underline{A}}_{kl}hw'' = h(t_{kl}w'') = h(t^e_{in}w)$. If $t^e_{in}w = x$, then either $w = x$ or $(\exists l)w = t_{il}x$. If $w = x$, then $(i,n) \in E^e(w) = E^{\underline{A}}(hw)$, so $h(t^e_{in}x) = hx = t^{\underline{A}}_{in}hx$. If $w = t_{il}x$, then $(i,n) \in E^e(x)\setminus\mathrm{Id}$ and thus $hx = t^{\underline{A}}_{in}h(t_{il}x)$ (since $h(t_{il}x)T^{\underline{A}}_i hx$). ■(3)

(4) $\quad f(w,i,\sigma) = t_{i\sigma}w''$, $\quad w'' \in D_m$ and $hwT^{\underline{A}}_i hw''$, or
$\quad\quad f(w,i,\sigma) = t_{ik}w''$, $\quad w'' \in D_m$ and $b(w,i,\sigma) = t^{\underline{A}}_{ik}hw''$

and

(5) $\quad f(w,i,\sigma) \notin D_m$.

Indeed, let $(w,i,\sigma) \in H_m$ and $\underline{f} \stackrel{\text{def}}{=} f(w,i,\sigma)$, $\underline{b} \stackrel{\text{def}}{=} b(w,i,\sigma)$. Now $\underline{f} \in \{t^e_{in}w : n \in \alpha\cup\mathrm{Tm}\}$, thus $\underline{f} \in D_m$ implies $h\underline{f} \notin \sigma^{\underline{C}}[k]$ by the definition of H_m. Suppose that $E^{\underline{A}}(\underline{b})^\star\{i\} = \{i\}$. Then $\underline{f} = t^e_{i\sigma}w$. It is easily seen from the definition of $t^e_{i\sigma}w$ that $t^e_{i\sigma}w = t^e_{i\sigma}w''$, where either $w'' = w$ or $(\exists l)w = t_{il}w''$. In both cases we have $w'' \in D_m$ and $hwT^{\underline{A}}_i hw''$ by $*7$. Moreover, $\underline{f} = t_{i\sigma}w'' \notin D_m$ by $*6$. Now suppose that $E^{\underline{A}}(\underline{b})^\star\{i\} \neq \{i\}$. Then $\underline{f} = t^e_{ik}w$, where $k = \min E^{\underline{A}}(\underline{b})^\star\{i\} \setminus \{i\}$. That is, $k \neq i$ and $\underline{b} \in \mathrm{d}^{\underline{A}}_{ik}$. Then $\underline{b} = t^{\underline{A}}_{ik}hw$ since $hwT^{\underline{A}}_i\underline{b}$. Thus $\underline{f} = t^e_{ik}w \notin D_m$, since $t^e_{ik}w \in D_m$ would imply $h(t^e_{ik}w) = t^{\underline{A}}_{ik}hw = \underline{b} \in \sigma^{\underline{C}}[k]$ by (3). This, together with $k = \min E^{\underline{A}}(hw)^\star\{k\}$ (since $E^{\underline{A}}(\underline{b}) = E^{\underline{A}}(hw)(i/k)$) implies $t^e_{ik}w = t_{ik}w''$ by $*5$, where either $w'' = w$ or $(\exists l)w = t_{il}w''$, so in both cases we have $w'' \in D_m$ and $hwT^{\underline{A}}_i hw''$, that is, $\underline{b} = t^{\underline{A}}_{ik}hw''$. ■(4),(5).

Now we are ready to prove that h_{m+1} is a function. We have to show that $(w, \underline{f})$, $(w, g) \in h_{m+1}$ imply $\underline{f} = g$. Suppose that $(w, \underline{f})$, $(w, g) \in h_{m+1}$.
Case 1. $(w, \underline{f})$, $(w, g) \in h$. Then $\underline{f} = g$ by $*3$.
Case 2. $(w, \underline{f}) \in h$, $(w, g) \notin h$. Then $w \in D_m$, thus $(w, g) = (t^e_{in}w', t^{\underline{A}}_{in}hw')$ for some $n \in \alpha$ and $(w', i) \in G_m$ by (5). Since $t^e_{in}w' = w \in D_m$, (3) gives $\underline{f} = hw = h(t^e_{in}w') = t^{\underline{A}}_{in}hw' = g$.
Case 3. $(w, \underline{f}) \notin h$, $(w, g) \notin h$. It will suffice to show

$$f(w, i, \sigma) = f(w', i', \sigma) \Longrightarrow b(w, i, \sigma) = b(w', i', \sigma)$$

$$f(w, i, \sigma) = t^e_{jn}w' \Longrightarrow b(w, i, \sigma) = t^{\underline{A}}_{jn}hw'$$

$$t^e_{il}w = t^e_{jn}w' \Longrightarrow t^{\underline{A}}_{il}hw = t^{\underline{A}}_{jn}hw'$$

for all (w, i, σ), $(w', i', \sigma') \in H_m$, (w', j), $(w, i) \in G_m$ and l, $n \in \alpha$. Suppose that $\underline{f} \stackrel{\text{def}}{=} f(w, i, \sigma) = f(w', i', \sigma')$. By (4), either $\underline{f} = t_{i\sigma}w''$, $i = i'$, $\sigma = \sigma'$ and $hwT^{\underline{A}}_i hw''T^{\underline{A}}_i hw'$, and in this case we have $b(w, i, \sigma) = b(w', i, \sigma) = b(w', i', \sigma')$ because of the choice of the function b; or $\underline{f} = t_{ik}w''$ and then $b(w, i, \sigma) = t^{\underline{A}}_{ik}hw'' = b(w', i', \sigma')$. Suppose that $f(w, i, \sigma) = t^e_{jn}w' = t_{ik}w''$. Then $t^e_{jn}w' \notin D_m$ by (5), and thus, since $w' \in D_m$ and $n \in \alpha$, the defnition of $t^e_{jn}w'$ shows that $k \in \alpha$. Then $b(w, i, \sigma) = t^{\underline{A}}_{ik}hw'' = t^{\underline{A}}_{jn}hw'$ by (4) and (2). Suppose that $t^e_{il}w = t^e_{jn}w' = \underline{f}$. If $\underline{f} \in D_m$, then (3) gives $t^{\underline{A}}_{il}hw = t^{\underline{A}}_{jn}hw'$. If $\underline{f} \notin D_m$, then $\underline{f}$ is of the form $t_{pk}w''$, and then $t^{\underline{A}}_{il}hw = t^{\underline{A}}_{pk}hw'' = t^{\underline{A}}_{jn}hw'$ by (2).
With this we have shown that h_{m+1} is a function.

Now we show that h_{m+1} satisfies $*5$. Let $w \in D_{m+1}$. If $w \in D_m$ then we are finished since $h_m \subseteq h_{m+1}$, and h_m satisfies $*5$. So suppose that $w \in W \setminus D_m$. Then $w = t_{ik}w''$ for some $i \in \alpha$, $k \in \alpha \cup \mathrm{Tm}$ and $w'' \in D_m$; and $E^e(w) = E^e(w'')(i/k) = E^{\underline{A}}(hw'')(i/k)$. Suppose that $w = f(w', i, \sigma)$ for some $(w', i, \sigma) \in H_m$. Let $b \stackrel{\text{def}}{=} b(w', i, \sigma)$. If $k \notin \alpha$, then $k = \sigma$ by (4) and $E^{\underline{A}}(b) = E^{\underline{A}}(hw'')(i/\sigma)$ since $bT^{\underline{A}}_i hw''$ and $E^{\underline{A}}(b)^\star\{i\} = \{i\}$. If $k \in \alpha$, then $E^{\underline{A}}(b) = E^{\underline{A}}(hw'')(i/k)$ since $b = t^{\underline{A}}_{ik}hw''$. Suppose that $w = t^e_{jn}w'$ for some

$n \in \alpha$ and $(w', j) \in G_m$. Then $\underline{f} \stackrel{\text{def}}{=} h_{m+1}w = t^{\underline{A}}_{jn}hw' = t^{\underline{A}}_{ik}hw''$ by (2), whence $E^{\underline{A}}(\underline{f}) = E^{\underline{A}}(hw'')(i/k)$. Thus we have shown that h_{m+1} satisfies $*5$.

P_{m+1} satisfies $*4$ since $*5$ is satisfied and P_m satisfies $*4$. As an illustration we consider two cases. Let $(w, \mathrm{d}_{ij}) \in P_{m+1}$, $w \in W$, $(i,j) \in E^e(w)$. Then $(i,j) \in E^{\underline{A}}(h_{m+1}w)$ by $*5$, so $h_{m+1}w \in \mathrm{d}^{\underline{C}}_{ij}$. Let $(t^e_{in}w, -\sigma) \in P_{m+1}$, where $(w, -\mathrm{c}_i\sigma) \in P_m$ and $n \in \alpha$. Then $h_m w \in (-\mathrm{c}_i\sigma)^{\underline{C}}[k]$, and since, as is easily checked, $h_m w T^{\underline{A}}_i h_{m+1}(t^e_{in}w)$, we get $h_{m+1}(t^e_{in}w) \in (-\sigma)^{\underline{C}}[k]$. The remaining cases are completely analogous, so we omit the details.

To show that h_{m+1} satisfies $*6$ suppose that $t_{i\sigma}w \in D_{m+1} \setminus D_m$. Then $t_{i\sigma}w = f(w', i, \sigma)$ for some $(w', i, \sigma) \in H_m$ (see the proof of (4) and (5)), and then $b(w', i, \sigma) \in \sigma^{\underline{C}}[k]$.

To show that h_{m+1} satisfies $*7$ suppose that $t_{in}w \in D_{m+1} \setminus D_m$. If $t_{in}w = f(w', i, \sigma)$, then $h_{m+1}(t_{in}w)T^{\underline{A}}_i hw$ by (4). If $t_{in}w = t^e_{jk}w'$, then $h_{m+1}(t_{in}w)T^{\underline{A}}_i hw$ by (2).

With this we have estabilished $(*)$ for P_{m+1} and h_{m+1}.

Let $P \stackrel{\text{def}}{=} \cup\{P_m : m < ||\tau||\}$. We show that P is a τ-tree. In what follows, we refer to conditions 1, 2(a)...2(c3) of the definition of τ-trees. (1) is satisfied by $*1$ and $*2$, and 2(a) is satisfied since $(x, \tau) \in P_0 \subseteq P$. One proves by induction that all P_ns and thus P satisfies 2(c1). Let $h \stackrel{\text{def}}{=} \cup\{h_m : m < ||\tau||\}$. Then $h : \mathrm{Dom}P \longrightarrow A$ and $hw \in \sigma^{\underline{C}}[k]$ if $(w, \sigma) \in P$, by $*3$ and $*4$, so 2(c3) and 2(b) is satisfied (since 2(c1) is satisfied). Condition 2(c2) is satisfied because of $*2$ (and the definition of P_{m+1}). Thus we have shown that P is a τ-tree. For the intuitive meaning of this part of the proof, see Remark 3.4(ii) below.

(II) Proof of "there is a τ-tree $\Longrightarrow \{\underline{A} \in NCA_\alpha : |A| \leq N(\tau)\} \not\models \tau = 0$" (where $N(\tau)$ will only be defined later, at the end of the proof).

Let $e \in \mathrm{Eqrel}(\alpha)$. By induction, define

$$N^e_0 \stackrel{\text{def}}{=} \{x\},$$

$$N^e_{m+1} \stackrel{\text{def}}{=} \{t^e_{in}w : w \in N^e_m, i \in \alpha, n \in \alpha \cup \mathrm{Tm}\},$$

$$N^e \stackrel{\text{def}}{=} \cup\{N^e_m : m \in \omega\}.$$

For i, $j \in \alpha$ let

$$E_{ij} \stackrel{\text{def}}{=} \{w \in N^e : (i,j) \in E^e(w)\},$$

$$T_i \stackrel{\text{def}}{=} \{(w,w) : w \in N^e\} \cup \{(w, t_{in}w) \in {}^2(N^e) : n \in \alpha \cup \mathrm{Tm}\} \cup \{(t_{in}w, w) \in {}^2(N^e) : n \in \alpha \cup \mathrm{Tm}\} \cup \{(t_{in}w, t_{im}w) \in {}^2(N^e) : n, m \in \alpha \cup \mathrm{Tm}\}.$$

$$\underline{N}^e \stackrel{\text{def}}{=} \langle N^e, T_i, E_{ij}\rangle_{i,j \in \alpha}.$$

LEMMA 3.1 $\underline{N}^e \in NAt_\alpha$.

Proof: Fix $e \in \mathrm{Eqrel}(\alpha)$. Let $n \in \omega$ and $N^n \stackrel{\mathrm{def}}{=} \cup\{N^e_m : m \leq n\}$. We claim that

(1) $N^n = \mathrm{Subterm}(N^n)$.

(2) If $t_{ik}w \in N^e$ and $k \in \alpha$ then (i)...(iii) below hold.

(i) $(i,k) \notin E^e(w)$

(ii) $k = \min(E^e(w)^\bigstar\{k\})$

(iii) w does not start with i.

Indeed, (1) and (2) are easily proved by induction using the definition of $t^e_{in}w$. Suppose that (1) holds for N^m and (2) holds for all $w' \in N^m$. Let $w \in N^m$, $i \in \alpha$, and $n \in \alpha \cup \mathrm{Tm}$. Let $k \stackrel{\mathrm{def}}{=} \min(E^e(w)^\bigstar\{n\})$ if $n \in \alpha$, and $k \stackrel{\mathrm{def}}{=} n$ if $n \notin \alpha$. Then it is immediate from the definition of $t^e_{in}w$ that (1) holds for N^{m+1}. Suppose that $n \in \alpha$. If $t^e_{in}w \in \{w, w'\} \subseteq N^m$, then we are done. Suppose that $t^e_{in}w = t_{ik}w$. Then from $(i,n) \notin E^e(w)$ and $(n,k) \in E^e(w)$ we conclude that $(i,k) \notin E^e(w)$ and $k = \min(E^e(w)^\bigstar\{k\})$, moreover, w does not start with i. Suppose that $t^e_{in}w = t_{ik}w'$. Then w' does not start with i, since $w = t_{il}w' \in N^m$, furthermore, $(i,k) \notin E^e(w')$ and $(i,n) \notin E^e(w)$, $(n,k) \in E^e(w)$ gives $(l,k) \notin E^e(w')$, thus $E^e(w)^\bigstar\{n\} = E^e(w)^\bigstar\{k\} = E^e(w')^\bigstar\{k\}$, so $k = \min(E^e(w')^\bigstar\{k\})$. This proves (1) and (2).

Now we return to the proof of $\underline{N}^e \in NAt_\alpha$. Let $i, j, k \in \alpha$. T_i is obviously reflexive and symmetric, and its transitivity is a consequence of (2)(iii). Moreover, $E_{ii} = N^e$, $E_{ij} = E_{ji}$ and $E_{ik} \cap E_{kj} \subseteq E_{ij}$, since $E^e(w) \in \mathrm{Eqrel}(\alpha)$ for all $w \in N^e$. Suppose that $k \notin \{i,j\}$, and let wT_kz. It is clear from the definition of T_k that $E^e(w) \restriction (\alpha \setminus \{k\}) = E^e(z) \restriction (\alpha \setminus \{k\})$, thus $w \in E_{ij}$ implies $z \in E_{ij}$. Suppose that $i \neq j$. We show that $T_i \cap {}^2E_{ij} \subseteq \mathrm{Id}$. Let wT_iz, $w \in E_{ij}$, $w \neq z$. If $z = t_{in}w$ and $n \in \alpha$, then $(i,n) \notin E^e(w)$ by (2)(i), and $(i,n) \in E^e(z)$, thus $(i,j) \notin E^e(z)$ (since $(i,j) \in E^e(w)$); if $n \notin \alpha$, then i is singular in $E^e(z)$, so $(i,j) \notin E^e(z)$; thus in both cases, we have $z \notin E_{ij}$. If $w = t_{in}z$ for some $n \in \alpha \cup \mathrm{Tm}$, then $n \in \alpha$, since i is not singular in $E^e(w)$, and $(i,n) \notin E^e(z)$ by (2)(i), $(i,n) \in E^e(w)$, so $(i,j) \notin E^e(z)$. Suppose that $w = t_{in}w'$, $z = t_{im}w'$, for some n, $m \in \alpha \cup \mathrm{Tm}$. Then $n \in \alpha$, since i is not singular in $E^e(w)$. If $m \notin \alpha$, then i is singular in $E^e(z)$, thus $z \notin E_{ij}$, and we are finished. Suppose that $m \in \alpha$. Then $n = \min(E^e(w')^\bigstar\{n\})$ and $m = \min(E^e(w')^\bigstar\{m\})$ by (2)(ii). Since $w \neq z$, we have $n \neq m$, and thus $(n,m) \notin E^e(w')$. Then $(i,j) \in E^e(w')(i/n)$ gives $(i,j) \notin E^e(w')(i/m) = E^e(z)$, so $z \notin E_{ij}$. We have thus shown that $T_i \cap {}^2E_{ij} \subseteq \mathrm{Id}$.

So far we have proved $\underline{N}^e \in pNAt_\alpha$. Though this is all we need in the sequel, we prove $\underline{N}^e \in NAt_\alpha$ for completeness' sake. We have to show that $E_{ij} \subseteq T^\bigstar_k(E_{ik} \cap E_{kj})$. Let $w \in E_{ij}$. If $k \in \{i,j\}$, then we are finished. Suppose that $k \notin \{i,j\}$. Let $z \stackrel{\mathrm{def}}{=} t^e_{ki}w$. Then $z \in E_{ij} \cap E_{ki} = E_{ik} \cap E_{kj}$, and $wT_kt^e_{ki}w$, as can be seen from the definition of $t^e_{ki}w$. This completes the proof of $\underline{N}^e \in NAt_\alpha$. ■(Lemma 3.1)

REMARK 3.2 (i) The name "N^e" refers to "normal-form". This is because if we define the function $n^e : \mathrm{Tm}_X(t_\alpha) \longrightarrow \mathrm{Tm}_X(t_\alpha)$ as in Remark 3.1, then $N^e = \{w \in \mathrm{Tm}(t_\alpha) : n^e(w) = w\} = \mathrm{Rng}(n^e)$, that is, N^e is the set of expressions in normal form.

(ii) Let $\underline{N} \stackrel{\mathrm{def}}{=} \underline{N}^e \upharpoonright \mathrm{Tm}(t'_\alpha)$. Then $\underline{N}$ is the free NAt_α with defining relation $E(x) = e$ "generated" by one element, in the following sense: Let $\underline{A} \in NAt_\alpha$, $a \in A$ and $E^{\underline{A}}(a) = e$. Then there is a function $k : \underline{N} \longrightarrow \underline{A}$ such that $k(x) = a$, and for all $\underline{B} \in NAt_\alpha$, $b \in B$, $E^{\underline{B}}(b) = e$, and if $h : \underline{B} \longrightarrow \underline{A}$, $h(b) = a$, then there is a function $f : \underline{N} \longrightarrow \underline{B}$ with $f(x) = b$ and $k = h \circ f$. (The second part of the statement says that k is the "smallest".) More generally, the following is true: Let $e \in \mathrm{Eqrel}(\alpha)$, $\underline{A} \in NAt_\alpha$, $a \in A$ and $E^{\underline{A}}(a) = e$. Then there is a function $k : \underline{N}^e \longrightarrow \underline{A}$ such that $k(x) = a$. ■

Let P be a τ-tree. Let $e \stackrel{\mathrm{def}}{=} \{(i,j) \in {}^2\alpha : (x, \mathrm{d}_{ij}) \in P\}$, and let $P' \stackrel{\mathrm{def}}{=} \{(w,\sigma) \in P : w \in N^e\}$. Then it is easy to check that P' is a τ-tree, and obviously $\mathrm{Dom}P' \subseteq N^e$. Let $\underline{B}(P) \stackrel{\mathrm{def}}{=} \underline{N}^e \upharpoonright \mathrm{Dom}P'$. For all $i \in \alpha$, let $K_i \stackrel{\mathrm{def}}{=} \{w \in \mathrm{Dom}P' : (\exists\sigma)(w, -\mathrm{c}_i\sigma) \in P'\}$.

LEMMA 3.2 *Let P be a τ-tree, and $\underline{B} \stackrel{\mathrm{def}}{=} \underline{B}(P)$. Then*

(i) *$\underline{B} \in pNAt_\alpha$, and $|B| \leq \beta \stackrel{\mathrm{def}}{=} (||\tau||) \cdot (|\alpha| \cdot (|\alpha| + |\mathrm{Subterm}(\tau)'|))^{||\tau||}$.*

(ii) *If $w \in K_i$, then $(\forall j \in \alpha)(t^{\underline{B}}_{ij} w$ exists).*

(iii) *If $\underline{B} \subseteq \underline{A} \in NAt_\alpha$, and $(\forall i \in \alpha)(\forall w \in K_i)(T_i^{\underline{B}^*}\{w\} = T_i^{\underline{A}^*}\{w\})$, then $\underline{CmA} \not\models \tau = 0$.*

Proof: (i) follows from Lemma 3.1 and from $|\{w \in \mathrm{Tm}(t''_\alpha) : ||w|| \leq ||\tau||\}| \leq \beta$, where $t''_\alpha = \{(t_{in}, 1) : i \in \alpha, n \in \alpha \cup \mathrm{Subterm}(\tau)'\}$. (ii) follows from P' being a τ-tree, since $(\forall w \in N^e) t^e_{in} w = t^{\underline{N}^e}_{in} w$. To prove (iii), suppose that $\underline{A} \supseteq \underline{B}$ satisfies the premiss of (iii). For all $y \in X$, let $k(y) \stackrel{\mathrm{def}}{=} \{w \in B : (w,y) \in P'\}$. Then $k : X \longrightarrow \mathrm{Sb}A$. Let $\underline{C} \stackrel{\mathrm{def}}{=} \underline{CmA}$. We prove $(w,\sigma) \in P' \Rightarrow w \in \sigma^{\underline{C}}[k]$ by induction on σ. Let $(*)$ be the following statement:

$$(*) \qquad (\forall w \in B)[(w,\sigma) \in P' \Rightarrow w \in \sigma^{\underline{C}}[k]]$$

If $\sigma \in X$, then $(*)$ holds by the definition of k. If $(w, \mathrm{d}_{ij}) \in P'$, then $(i,j) \in E^e(w)$, thus $w \in \mathrm{d}^{\underline{C}}_{ij}$. Suppose that $(*)$ holds for all elements of $\mathrm{Subterm}(\sigma \cdot \delta)'$. This implies that if $(w, \sigma \cdot \delta) \in P'$, then (w,σ), $(w,\delta) \in P'$, and thus $w \in \sigma^{\underline{C}}[k] \cap \delta^{\underline{C}}[k] = (\sigma \cdot \delta)^{\underline{C}}[k]$. Suppose that $(w, -\sigma) \in P'$. If $\sigma = y \in X$, then $(w, -y) \in P'$ implies $(w,y) \notin P'$, thus $w \notin k(y)$, that is, $w \in (-y)^{\underline{C}}[k]$. If $\sigma = \mathrm{d}_{ij}$, then $(w, -\mathrm{d}_{ij}) \in P'$ gives $(i,j) \notin E^e(w)$, thus $w \in (-\mathrm{d}_{ij})^{\underline{C}}$. If $\sigma = \sigma' \cdot \delta$, then it follows from $(w, -(\sigma' \cdot \delta)) \in P'$ that either $(w, -\sigma') \in P'$, and then $w \in (-\sigma')^{\underline{C}}[k] \subseteq -(\sigma' \cdot \delta)^{\underline{C}}[k]$, or $(w, -\delta) \in P'$, and then $w \in -(\sigma' \cdot \delta)^{\underline{C}}[k]$ by the same argument. If $\sigma = -\sigma'$, then $(w, -(-\sigma')) \in P'$ implies $(w, \sigma') \in P'$, and thus $w \in (\sigma')^{\underline{C}}[k] = (-(-\sigma'))^{\underline{C}}[k]$. If $\sigma = c_i\sigma'$, then it follows from $(w, -c_i\sigma') \in P'$ that $w \in K_i$, and $(z, -\sigma') \in P'$ for all $z \in T_i^{\underline{B}^\star}\{w\}$, thus $z \notin (\sigma')^{\underline{C}}[k]$, but $T_i^{\underline{B}^\star}\{w\} = T_i^{\underline{A}^\star}\{w\}$ by assumption, so $w \in (-c_i\sigma')^{\underline{C}}[k]$. If $(w, c_i\sigma) \in P'$, then $(t_{in}^e w, \sigma) \in P'$ for some n, and then $t_{in}^e w \in \sigma^{\underline{C}}[k]$ and $w T_i^{\underline{A}} t_{in}^e w$, thus $w \in (c_i\sigma)^{\underline{C}}[k]$. ■(Lemma 3.2)

REMARK 3.3 (i) Lemma 3.2 (iii) may suggest that if $a \in \tau^{\underline{C}}[k]$, where $\underline{C} = \underline{CmA}$, $\underline{A} \in NAt_\alpha$ and $k : X \longrightarrow \mathrm{Sb}A$, then there is a finite neighbourhood of a in $\underline{A}$ (its size being dependent on τ) which influences $a \in \tau^{\underline{C}}[k]$ and such that "things outside it" have no effect on a, and the τ-tree is really nothing but this neighbourhood in an abstract form. Yet this holds only for "free" $\underline{N}^e \in NAt_\alpha$'s and not generally: There is a $\tau \in \mathrm{Tm}$ and $\underline{A} \in NAt_\alpha$ such that $\underline{CmA} \not\models \tau = 0$ but $(\forall B \subseteq_\omega A)[\underline{Cm}(\underline{A} \upharpoonright B) \models \tau = 0]$ while $(\forall X \subseteq_\omega A)(\exists B \subseteq_\omega A)[X \subseteq B$ and $\underline{A} \upharpoonright B \in NAt_\alpha]$. An example for such a τ and $\underline{A}$ is the τ given in Remark 3.4 (ii) and the $\underline{At}(V) \in NAt_\alpha$ constructed from the Crs_α-unit V given immediately after it.
(ii) It is fairly easy to construct a $\underline{B}(P) \subseteq \underline{A} \in NAt_\alpha$ satisfying the condition of Lemma 3.2 (iii): $\underline{B}(P)$ can be extended essentially "freely" to an NAt_α. But this extension will be infinite. Next we show that there is a finite extension provided $\alpha < \omega$. ■

LEMMA 3.3 *Let $\underline{B} \in pNAt_\alpha$ be arbitrary. Then there is an $\underline{A} \in NAt_\alpha$ satisfying (i)... (iii) below.*

(i) $\underline{B} \subseteq A$.

(ii) $|A| \leq \eta \cdot |B|$, *where* $\eta \stackrel{\text{def}}{=} 2^{|\alpha| \cdot |\mathrm{Eqrel}(\alpha)|^2} \cdot |\mathrm{Eqrel}(\alpha)|$.

(iii) *If $b \in B$, $a \in A \setminus B$ and $b T_i^{\underline{A}} a$, then $(\exists j \in \alpha)[t_{ij}^{\underline{B}}$ does not exist$]$.*

Proof: First we introduce some notations. Let $\underline{B} \in pNAt_\alpha$, e, $e' \in \mathrm{Eqrel}(\alpha)$ and $i \in \alpha$. Then

(1) $\underline{B}(e) \stackrel{\text{def}}{=} \{w \in B : E^{\underline{B}}(w) = e\}$.

(2) We say that $e \rangle ie'$, if $(\exists k \in \alpha) e' = e(i/k)$; and $e \stackrel{i}{\longleftrightarrow} e'$ means $(e)ie'$ and $e')ie)$. $e \rangle ie'$ is said to be *good in* $\underline{B}$ if $\underline{B}(e) \times \underline{B}(e') \cap T_i^{\underline{B}} : \underline{B}(e) \longrightarrow \underline{B}(e')$; and $e \stackrel{i}{\longleftrightarrow} e'$ is *good in* $\underline{B}$ if $\underline{B}(e) \times \underline{B}(e') \cap T_i^{\underline{B}} : \underline{B}(e) \rightarrowtail\!\!\!\!\twoheadrightarrow \underline{B}(e')$.

(3) Let $H \stackrel{\text{def}}{=} \{(e, e', i) : e \rangle ie', e \neq e'\}$, $G \stackrel{\text{def}}{=} \{(e, e', i) : e \stackrel{i}{\longleftrightarrow} e', e \neq e'\}$.

(4) $\underline{A} \supseteq \underline{B}$ is said to be a *good extension of* $\underline{B}$ if it satisfies the condition formulated in Lemma 3.3 (iii).

Claim 3.1 *Let $\underline{B} \in pNAt_\alpha$ and suppose that $e\rangle ie'$ is good in $\underline{B}$, for all $(e, e', i) \in H$. Then $\underline{B} \in NAt_\alpha$.*

Proof: Let $\underline{B} = \langle B, T_i, E_{ij}\rangle_{i,j\in\alpha} \in pNAt_\alpha$ be as in the statement. We have to check $E_{ij} \subseteq T_k^\bigstar(E_{ik} \cap E_{kj})$ for all i, j, $k \in \alpha$. Let $b \in E_{ij}$. Then $b \in \underline{B}(e)$, where $e = E^{\underline{B}}(b)$, and $(i, j) \in e$. Let $e' \stackrel{\text{def}}{=} e(k/i)$. We may suppose that $e \neq e'$. Then $e\rangle ke'$ is good in $\underline{B}$. Thus there is an $a \in \underline{B}(e')$ with bT_ka. Since $(i, j) \in e$, we have (i, k), $(j, k) \in e'$, that is, $a \in E_{ik} \cap E_{kj}$. ■(Claim 3.1)

Claim 3.1 makes it possible to extend a $\underline{B} \in pNAt_\alpha$ to an $\underline{A} \in NAt_\alpha$ by "repairing" the $e\rangle ie'$'s step by step.

An $\underline{N} \in pNAt_\alpha$ is said to be *regular* , if

1. $N = W \times \text{Eqrel}(\alpha)$ for some set W, and
2. $\underline{N}(e) = W \times \{e\}$ for all $e \in \text{Eqrel}(\alpha)$.

Claim 3.2 *Suppose that $\underline{N} \in pNAt_\alpha$ is regular, and $e \stackrel{i}{\longleftrightarrow} e'$ is not good in $\underline{N}$, where $(e, e', i) \in G$. Then there is a regular $\underline{M} \in pNAt_\alpha$ such that*

(1) $\underline{N} \subseteq \underline{M}$, $|M| = 2 \cdot |N|$

(2) *$e \stackrel{i}{\longleftrightarrow} e'$ is good in $\underline{M}$*

(3) *if $e_0 \stackrel{j}{\longleftrightarrow} e_1$ is good in $\underline{N}$, then $e_0 \stackrel{j}{\longleftrightarrow} e_1$ is good in $\underline{M}$, for all $(e_0, e_1, j) \in G$.*

(4) *$\underline{M}$ is a good extension of $\underline{N}$.*

Proof: Suppose that $N = W \times \text{Eqrel}(\alpha)$. Let $W' \stackrel{\text{def}}{=} W \cup (1 \times W)$ and $M \stackrel{\text{def}}{=} W' \times \text{Eqrel}(\alpha)$ (cf. Fig. 1).

Let

$$E_{ij}^{\underline{M}} \stackrel{\text{def}}{=} \{(w, \bar{e}) \in M : (i, j) \in \bar{e}\}$$

for all i, $j \in \alpha$. Then $\underline{M}(\bar{e}) = W' \times \{\bar{e}\}$ for all $\bar{e} \in \text{Eqrel}(\alpha)$.

Since $e \stackrel{i}{\longleftrightarrow} e'$ is not good in $\underline{N}$, either $e\rangle ie'$ or $e'\rangle ie$ is not good in $\underline{N}$. We may suppose that the former case applies. Let

$D \stackrel{\text{def}}{=} \{w \in \underline{N}(e) : (\neg\exists z \in \underline{N}(e'))wT_i^{\underline{N}}z\}$, and let g be such that

$g : \underline{M}(e) \rightarrowtail\!\!\!\!\twoheadrightarrow \underline{M}(e')$,

$h \subseteq g$, where $h \stackrel{\text{def}}{=} \underline{N}(e) \times \underline{N}(e') \cap T_i^{\underline{N}}$, and

$g^\bigstar D \subseteq (1 \times W) \times \{e'\}$. (See Fig. 1.)

There is such a function g, since both h and h^{-1} are functions because of $\underline{N} \in pNAt_\alpha$ and $(e, e', i) \in G$. For all $j \in \alpha$, let

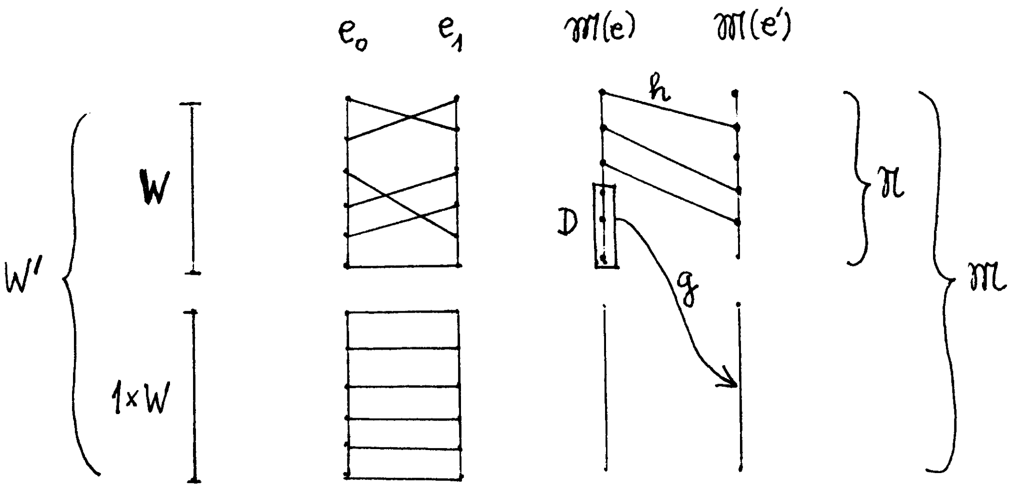


Figure 1.

$\bar{g} \stackrel{\text{def}}{=} g \cup g^{-1}$,

$R_j \stackrel{\text{def}}{=} \{\langle (w, e_0), (w, e_1)\rangle : w \in 1{\times}W, (e_0, e_1, j) \in G, e_0 \stackrel{j}{\longleftrightarrow} e_1 \text{ is good in } \underline{N}\} \cup \mathrm{Id}_{(1\times W)}$,

$L_j \stackrel{\text{def}}{=} T_j^{\underline{N}} \cup R_j$ if $j \neq i$, and

$L_i \stackrel{\text{def}}{=} T_i^{\underline{N}} \cup R_i \cup \bar{g}$,

$T_j^{\underline{M}} \stackrel{\text{def}}{=}$ "the transitive closure of L_j",

$\underline{M} \stackrel{\text{def}}{=} \langle M, T_k^{\underline{M}}, E_{kl}^{\underline{M}}\rangle_{k,l\in\alpha}$.

We will show that $\underline{M}$ has the desired properties. To this end, let us first describe the relations $T_j^{\underline{M}}$ a bit more explicitly.

(1) Let $j \in \alpha$ and $pT_j^{\underline{M}}q$. Then one of the following cases holds:

(i) p, $q \in N$, and $pT_j^{\underline{N}}q$

(ii) $p \in N$, $q \notin N$, $j = i$, and $pT_i^{\underline{N}}a\bar{g}bR_iq$ for some a, $b \in M$

(iii) $p \notin N$, $q \in N$, $j = i$, and $pR_ia\bar{g}bT_i^{\underline{N}}q$ for some a, $b \in M$

(iv) $p \notin N$, $q \notin N$, $j = i$, and pR_jq

(v) $p \notin N$, $q \notin N$, $j = i$, and $pR_ia\bar{g}bR_iq$ for some a, $b \in M \setminus N$.

Proof of (1): It will suffice to show that if $pT_j^{\underline{M}}qL_jr$, and p, q satisfies one of (i)...(v), then p, r satisfies one of (i)...(v). (If $p = q$, then clearly (i) or (iv) holds.) The nontrivial cases are as follows (we note that R_j is transitive, and $e \neq e'$):

$$\text{(a)} \qquad pT_i^{\underline{N}}a\bar{g}bR_iq\bar{g}r$$

We claim that in this case we have $r = a$, and thus (i) holds. bR_iq gives $b = (w, e_0)$ and $q = (w, e_1)$ for some w, e_0 and e_1, where either $e_0 \overset{i}{\longleftrightarrow} e_1$ is good in $\underline{N}$, or $b = q$. We have $\{e_0, e_1\} \subseteq \{e, e'\}$ by $a\bar{g}b$, $q\bar{g}r$, and thus, since $e \overset{i}{\longleftrightarrow} e'$ is not good in $\underline{N}$, we get $e_0 = e_1$, whence $b = q$. But then $a\bar{g}b$, $b\bar{g}r$ gives $a = r$.

$$\text{(b)} \qquad pR_ia\bar{g}bT_i^{\underline{N}}q\bar{g}r \text{ and } r \notin N$$

We show that $r = a$, and thus (iv) holds for p, r. It follows from $a\bar{g}b$ and $q\bar{g}r$ that $b = (w, e_0)$ and $q = (z, e_1)$ for some w, z and $\{e_0, e_1\} \subseteq \{e, e'\}$. Then $a \notin N$, $b \in N$, $q \in N$, $r \notin N$, $bT_i^{\underline{N}}q$, and $e_0 = e_1$ by the definition of $\bar{g}$. But then bT_iq gives $b = q$, since i is not singular in e or e'. This, together with $a\bar{g}b$, $b\bar{g}r$, implies $a = r$.

$$\text{(c)} \qquad pR_ia\bar{g}bR_iq\bar{g}r$$

As in the previous cases, $a\bar{g}bR_iq\bar{g}r$ gives $b = q$, and thus $a = r$, so case (iv) holds for p, r. ■(1)

Now we are ready to prove that $\underline{M}$ has the desired properties. First we show that $\underline{M} \in pNAt_\alpha$. Clause (i) in Definition 3.3 is satisfied since the relations L_j are reflexive and symmetric; and (ii) is obviously satisfied. Let $k \notin \{n, m\}$, $k, n, m \in \alpha$, $p \in E_{nm}^{\underline{M}}$ and $pT_k^{\underline{M}}q$. Then $q \in E_{nm}^{\underline{M}}$, since L_k is easily seen to "preserve $E_{nm}^{\underline{M}}$". Let $n \neq m$, $p, q \in E_{nm}^{\underline{M}}$ and $pT_n^{\underline{M}}q$. We prove $p = q$ by examining cases (i) ... (v) of (1).

If (i) holds, then $p = q$, since $\underline{N} \in pNAt_\alpha$. Note that $E^{\underline{M}}p = E^{\underline{M}}q$, so $p = (w, \bar{e})$, $q = (z, \bar{e})$ for some $w, z, \bar{e}$, thus we have $p = q$ in case (iv), too. Now we show that there are no more possibilities. Suppose that $pT_i^{\underline{N}}a\bar{g}bR_iq$, $n = i$. Then $a \in N$, $b \notin N$, so $\{E^{\underline{N}}(a), E^{\underline{N}}(b)\} = \{e, e'\}$. Suppose that $E^{\underline{N}}(a) = e$ (the other case, $E^{\underline{N}}(a) = e'$, is completely similar). Then bR_iq, so $e' \overset{i}{\longleftrightarrow} \bar{e}$ is good in $\underline{N}$ (this is true even if $b = q$ and thus $e' = \bar{e}$, since i is not singular in e'), thus there is an $r \in E^{\underline{N}}(e')$ such that $pT_i^{\underline{N}}r$, and then $aT_i^{\underline{N}}r$, whence $a \in N \setminus D$, contradicting $a\bar{g}b$, $b \notin N$. Thus case (ii) cannot hold for p, q, and it can be shown similarly, that neither can (iii) or (v). This shows that $\underline{M} \in pNAt_\alpha$.

Clearly, $\underline{M}$ is regular, and $|M| = 2 \cdot |N|$. Now we show that $\underline{N} \subseteq \underline{M}$. Obviously, $E_{kl}^{\underline{N}} = N \cap E_{kl}^{\underline{M}}$ for all k, $l \in \alpha$. Let $j \in \alpha$, $p, q \in N$ and $pT_j^{\underline{M}}q$. Then $pT_j^{\underline{N}}q$ by (1). With this we have shown $\underline{N} \subseteq \underline{M}$. It is immediate from the construction of $\underline{M}$, that $e \stackrel{i}{\longleftrightarrow} e'$ is good in $\underline{M}$, and if $e_0 \stackrel{j}{\longleftrightarrow} e_1$ is good in $\underline{N}$, $(e_0, e_1, j) \in G$, then $e_0 \stackrel{j}{\longleftrightarrow} e_1$ is good in $\underline{M}$. To prove that $\underline{M}$ is a good extension of $\underline{N}$, suppose that $p \in N$, $q \in M \setminus N$, $j \in \alpha$, and $pT_j^{\underline{M}}q$. Then it follows from (1) that $j = i$ and there are a, b such that $pT_i^{\underline{N}} a\bar{g}bR_iq$. Then $t_{ik}^{\underline{N}}a$ does not exists, where $k \neq i$, and either $e' = e(i/k)$ or $e = e'(i/k)$. Since $pT_i^{\underline{N}}a$, we conclude that $t_{ik}^{\underline{N}}p$ does not exist either. ■(Claim 3.2)

Claim 3.3 *Let $\underline{N} \in pNAt_\alpha$ be regular and suppose that for all $(e_0, e_1, i) \in G$, $e_0 \stackrel{i}{\longleftrightarrow} e_1$ is good in $\underline{N}$. Then there is a regular $\underline{M} \in NAt_\alpha$ such that*

(1) $\underline{N} \subseteq \underline{M}$, $|M| = 2 \cdot |N|$,

(2) *$\underline{M}$ is a good extension of $\underline{N}$.*

Proof: The proof is similar to that of Claim 3.2. Let W,M,$E_{k,l}^{\underline{M}}$ and $\underline{M}(e)$ be the same as in the proof of Claim 3.2, that is, $N = W \times \mathrm{Eqrel}(\alpha)$, $M = [W \cup (1 \times W)] \times \mathrm{Eqrel}(\alpha)$, $E_{k,l}^{\underline{M}} = \{(w, \bar{e}) \in M : (k, l) \in \bar{e}\}$ and $\underline{M}(e) = \{(w, \bar{e}) \in M : \bar{e} = e\}$, if $k, l \in \alpha$, $e \in \mathrm{Eqrel}(\alpha)$. We may suppose that $\alpha \geq 2$ since $pNAt_\alpha = NAt_\alpha$ for $\alpha \leq 1$. Let $i \in \alpha$ and define $T_i^{\underline{M}}$ as follows: Let $j \in \alpha \setminus \{i\}$ be fixed. If i is singular in $e \in \mathrm{Eqrel}(\alpha)$, then let $g(e) : \underline{M}(e) \longrightarrow \underline{M}(e(i/j))$ be such that

$$h \subseteq g(e), \text{ where } h \stackrel{\text{def}}{=} T_i^{\underline{N}} \cap [\underline{N}(e) \times \underline{N}(e(i/j))],$$

$$(g(e) \setminus h)^\star M \subseteq M \setminus N,$$

$$\{\langle (w, e), (w, e(i/j)) \rangle : w \in 1 \times W\} \subseteq g(e), \text{ and}$$

$${}^2N \cap \ker(g(e)) = {}^2\underline{N}(e) \cap T_i^{\underline{N}} \text{ (see the picture).}$$

There *is* such a function $g(e)$ since h is a function. Let

$$g_i \stackrel{\text{def}}{=} \cup\{g(e) : e \in \mathrm{Eqrel}(\alpha), i \text{ is singular in } e\},$$

$$\bar{g}_i \stackrel{\text{def}}{=} g_i \cup g_i^{-1},$$

$$R_i \stackrel{\text{def}}{=} \{\langle (w, e), (w, e') \rangle : w \in 1 \times W, e \restriction (\alpha \setminus \{i\}) = e' \restriction (\alpha \setminus \{i\})\},$$

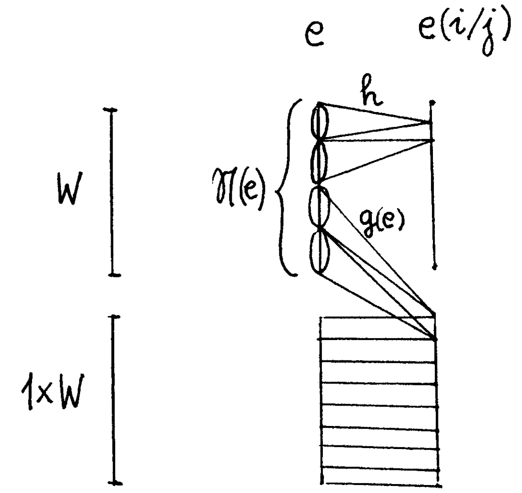
$$L_i \stackrel{\text{def}}{=} T_i^{\underline{N}} \cup \bar{g}_i \cup R_i,$$

$$T_i^{\underline{M}} \stackrel{\text{def}}{=} \text{“the transitive closure of } L_i\text{”},$$

$$\underline{M} \stackrel{\text{def}}{=} \langle M, T_i^{\underline{M}}, E_{i,j}^{\underline{M}} \rangle_{i,j \in \alpha}.$$

We want to show that $\underline{M}$ has the desired properties. Again, first we will have a closer look at the relations $T_i^{\underline{M}}$.

(1) Let $i \in \alpha$ and $pT_i^{\underline{M}}q$. Then one of the following cases holds:



(i) $pT_i^{\underline{N}}q$.

(ii) $pT_i^{\underline{N}}a\bar{g}_ibR_iq$ for some $a, b \in M$.

(iii) $pR_ia\bar{g}_ibT_i^{\underline{N}}q$ for some $a, b \in M$.

(iv) pR_iq.

The proof of (1) proceeds as in the previous claim. We show that if $pT_i^{\underline{M}}qL_ir$, and one of (i) ...(iv) holds for p, q, then one of (i) ...(iv) holds for p, r. The nontrivial cases are as follows.

$$\text{(a)} \qquad pT_i^{\underline{N}}a\bar{g}_ibR_iq\bar{g}_ir, \qquad r \in N.$$

Let $e \overset{\text{def}}{=} E^{\underline{N}}(a)$, $e' \overset{\text{def}}{=} E^{\underline{N}}(b)$. Then i is singular in e and $e' = e(i/j)$ (where $j \in \alpha \setminus \{i\}$ is the index "chosen" for i) since $a\bar{g}_ib$ and $a \in N$, $b \notin N$. By the same token i is singular in $E^{\underline{N}}(r)$ and $E^{\underline{N}}(q) = E^{\underline{N}}(r)(i/j)$. Now bR_iq gives $E^{\underline{N}}(q) \restriction (\alpha \setminus \{i\}) = e' \restriction (\alpha \setminus \{i\})$, so $E^{\underline{N}}(r) = e$ and $E^{\underline{N}}(q) = e'$. Then bR_iq gives $b = q$, whence $a\bar{g}_ib\bar{g}_ir$; and $a, r \in N$, $b \notin N$ gives $(a, r) \in \ker g_i$, thus $aT_i^{\underline{N}}r$, so $pT_i^{\underline{N}}r$. But this means that case (i) holds for p, r.

$$\text{(b)} \qquad pR_ia\bar{g}_ibT_i^{\underline{N}}q\bar{g}_ir, \qquad r \notin N.$$

Let $e = E^{\underline{N}}(b)$ and $e' = e(i/j)$. Reasoning as above, we get $E^{\underline{N}}(q) = e$. Then $a\bar{g}_i b$, $a \notin N$, $b \in N$ gives $bg(e)a$, and similarly $qg(e)r$, so $bT_i^{\underline{N}}q$ gives $a = r$, and we conclude that case (iv) obtains for p, r. ■(1)

Now we are ready to prove that $\underline{M}$ has the desired properties. The proof of $\underline{M} \in pNAt_\alpha$ is as in the previous claim, so we only give details for the (hardest) statement "$T_i^{\underline{M}} \cap {}^2E_{i,j}^{\underline{M}} \subseteq \mathrm{Id}$ if $i \neq j$".

Suppose that $pT_i^{\underline{M}}q$ and $p, q \in E_{ij}^{\underline{M}}$. Then $e'' \stackrel{\text{def}}{=} E^{\underline{M}}(p) = E^{\underline{M}}(q)$. Now one of (i)...(iv) (in (1) above) holds for p, q. If it is (i) or (iv) then we are done. Suppose that $pT_i^{\underline{N}}a\bar{g}_i bR_i q$. Let $e = E^{\underline{M}}(a)$, $e' = E^{\underline{M}}(b)$. Then i is singular in e but not in e' and e'', thus $e'' \stackrel{i}{\longleftrightarrow} e'$. Since $e'' \stackrel{i}{\longleftrightarrow} e'$ is good in $\underline{N}$, there is an $r \in E^{\underline{M}}(e')$ such that $pT_i^{\underline{N}}r$, that is, $aT_i^{\underline{N}}r$, contradicting $a\bar{g}_i b$, $b \notin N$. Thus (ii) cannot hold for p, q. A similar argument shows that (iii) cannot hold either. With this we have proved $\underline{M} \in pNAt_a$.

Let $(e, e', i) \in H$. If $(e, e', i) \in G$, then $e\rangle ie'$ is good in $\underline{M}$ since it was good in $\underline{N}$ and the construction preserves goodness. If $(e, e', i) \notin G$ then i is singular in e, and then the construction shows that $e\rangle ie'$ is good in $\underline{M}$. Thus $\underline{M} \in NAt_\alpha$ by Claim 3.1. The proof of the other properties are as in the previous claim. ■(Claim 3.3)

Now we begin the proof of Lemma 3.3. Let $\underline{B} \in pNAt_a$. Let $N \stackrel{\text{def}}{=} B \times \mathrm{Eqrel}(\alpha)$ and let $h : B \rightarrowtail N$ be such that $h^\star\underline{B}(e) \subseteq B \times \{e\}$ for all $e \in \mathrm{Eqrel}(\alpha)$. For all $i, j \in \alpha$ define

$$E_{ij}^{\underline{N}} \stackrel{\text{def}}{=} \{(b, e) \in N : (i, j) \in e\} \text{ and}$$

$$T_i^{\underline{N}} \stackrel{\text{def}}{=} \{(ha, hb) : (a, b) \in T_i^{\underline{B}}\} \cup \mathrm{Id} \restriction N,$$

$$\underline{N} \stackrel{\text{def}}{=} \langle N, T_i^{\underline{N}}, E_{ij}^{\underline{N}} \rangle_{i,j \in \alpha}.$$

It is easy to check that $h : \underline{B} \rightarrowtail \underline{N} \in pNAt_\alpha$, $\underline{N}$ is regular, $|N| = |B| \cdot |\mathrm{Eqrel}(\alpha)|$ and $\underline{N}$ is a good extension of $h^\star\underline{B}$. By repeated applications of Claim 3.2 and then using Claim 3.3 we get an $\underline{M} \in NAt_\alpha$ with $|M| \leq 2^{|H|} \cdot |N|$ which is a good extension of $\underline{N}$. From this structure $\underline{M}$ we obviously obtain an $\underline{A} \supseteq \underline{B}$ with the desired properties via isomorphism. ■(Lemma 3.3)

Let us return to the proof of "(II) There is a τ-tree $\Longrightarrow \{\underline{A} \in NCA_\alpha : |A| \leq N(\tau)\} \not\models \tau = 0$". Recall that just before Lemma 3.2 we have constructed a partial atomstructure $\underline{B} \in pNAt_\alpha$ from a given τ-tree P. Let $N(\tau) \stackrel{\text{def}}{=} 2^{\eta \cdot \beta}$, where η and β are as in Lemmas 3.3 and 3.2, respectively. Let $\underline{A} \in NAt_\alpha$ be an extension of $\underline{B}(P)$ given by Lemma 3.3. Then $|A| \leq \eta \cdot \beta$ and $\underline{C} \stackrel{\text{def}}{=} \underline{CmA} \not\models \tau = 0$ by Lemma 3.2. Clearly $|\underline{C}| \leq N(\tau)$. Thus $\{\underline{A} \in NCA_\alpha : |A| \leq N(\tau)\} \not\models \tau = 0$. If $\alpha < \omega$, then $N(\tau) \in \omega$ and in this case the function N is obviously computable. ■(Proposition 3.1)

Thus we have proved (i) and (ii) of Theorem 3.1 for $\alpha < \omega$: (i) follows from both Proposition 3.1(i) and Proposition 3.1(ii), since the existence of a τ-tree is easily seen to be decidable (if $\alpha < \omega$) and all equations can be (recursively) transformed into an equivalent equation of the form $\tau = 0$; and Theorem 3.1(ii) is an immediate corollary of Proposition 3.1(ii).

Next we turn to the proof of the decidability of EqNCA_α for $\alpha \geq \omega$. The proof proceeds by reducing this case to that of $\alpha < \omega$ *via* Lemma 3.4 below. Let ind(τ) be the set of indices occurring in τ, i.e. ind(τ) is the smallest $\gamma \subseteq \alpha$ such that $\tau \in \mathrm{Tm}_X(\mathrm{cyl}_\gamma)$. Note that ind$(\tau)$ is computable.

LEMMA 3.4 *Let* $\mathrm{ind}(\tau) \subset \beta \subseteq \alpha$. *Then*

$$NCA_\alpha \models \tau = 0 \Longleftrightarrow NCA_\beta \models \tau = 0.$$

Proof: We may suppose that α and β are ordinals. Suppose that $\underline{A} \in NCA_\alpha$, $\underline{A} \not\models \tau = 0$. Then $\underline{Rd_\beta A} \not\models \tau = 0$ and $\underline{Rd_\beta A} \in NCA_\beta$, thus $(NCA_\alpha \not\models \tau = 0 \Longrightarrow NCA_\beta \not\models \tau = 0)$. Note that α is a parameter of the concept of a τ-tree. Let us make this explicit and say that "P is a τ,α-tree" if P satisfies the conditions in Definition 3.2(iii). Suppose that $NCA_\beta \not\models \tau = 0$. Then there is a τ,β-tree P by Proposition 3.1(i). Let $P' \stackrel{\mathrm{def}}{=} \{(w,\sigma) \in P : (\forall i,n)[t_{in}$ occurs in $w \Rightarrow i \in \mathrm{ind}(\tau)]\}$. It is not hard to see that then P' is a τ,β-tree, too. Let $k \in \beta \setminus \mathrm{ind}(\tau)$ be arbitrary, $e \stackrel{\mathrm{def}}{=} \{(i,j) \in {}^2\beta : (x,\mathrm{d}_{ij}) \in P'\}$ and let $\bar{e}$ be the equivalence relation (on α) generated by the relation $e \cup \{k\} \times (\alpha \setminus \beta)$. Let

$$\begin{aligned} P'' \stackrel{\mathrm{def}}{=} \; & P' \cup \{(w,\mathrm{d}_{ij}) : (i,j) \in E^{\bar{e}}(w), w \in \mathrm{Dom}P'\} \\ & \cup \{(w,-\mathrm{d}_{ij}) : (i,j) \notin E^{\bar{e}}(w), w \in \mathrm{Dom}P', \quad i,j \in \alpha\}. \end{aligned}$$

We show that P'' is a τ,α-tree. Let $D \stackrel{\mathrm{def}}{=} \mathrm{Dom}P'' = \mathrm{Dom}P'$. It is not hard to show by induction that $E^e(w) = E^{\bar{e}}(w) \upharpoonright \beta$ and $t^e_{in}w = t^{\bar{e}}_{in}w$ for all $w \in D$, $i \in \beta$ and n. (This is where we use that α and β are ordinals.) From this it follows that condition (c3) in Definition 3.2(iii) is satisfied. The only remaining condition which is not trivially satisfied is

$$(w,-\mathrm{c}_i\sigma) \in P'' \Longrightarrow (\forall n \in \alpha \setminus \beta)(t^{\bar{e}}_{in}w,-\sigma) \in P''.$$

Since $k \notin \mathrm{ind}(\tau)$, for no l does t_{kl} or t_{nl} occur in w, whence $(k,n) \in E^{\bar{e}}(w)$ since $(k,n) \in \bar{e}$. Thus $t^{\bar{e}}_{in}w = t^{\bar{e}}_{ik}w$ and $(w,-\mathrm{c}_i\sigma) \in P'' \Rightarrow (w,-\mathrm{c}_i\sigma) \in P' \Rightarrow (t^e_{ik}w,-\sigma) \in P' \Rightarrow (t^e_{ik}w,-\sigma) \in P'' \Rightarrow (t^{\bar{e}}_{in}w,-\sigma) \in P''$ (since $t^e_{ik}w = t^{\bar{e}}_{ik}w = t^{\bar{e}}_{in}w$). With this we have shown that P'' is a τ,α-tree. From Proposition 3.1(i) it follows that $NCA_\alpha \not\models \tau = 0$. This completes the proof of $(NCA_\beta \not\models \tau = 0 \Longrightarrow NCA_\alpha \not\models \tau = 0)$. ■(Lemma 3.4)

Using Lemma 3.4 and Proposition 3.1 it is easy to construct an algorithm deciding EqNCA_α. (Note that the recursive function $N(\tau)$, the existence of which was proved in Proposition 3.1(ii), has α as a parameter, moreover in such a way that the two-place function $N(\tau,\alpha)$ is still computable.) Now we turn to the proof of Theorem 3.1(iii).

Proof of Theorem 3.1(iii): We may suppose $3 \subseteq \alpha$ without loss of generality. Define

$$\mathrm{t}^i_j x \stackrel{\mathrm{def}}{=} \mathrm{d}_{ij} \cdot \mathrm{c}_i x, \qquad \text{where } i, j \in \alpha$$

(this may be conceived of as a kind of dual of $\mathrm{s}^i_j x = \mathrm{c}_i(\mathrm{d}_{ij} \cdot x)$). Let

$$\tau x \stackrel{\mathrm{def}}{=} \mathrm{t}^1_0 \mathrm{t}^2_1 \mathrm{t}^0_2, \text{ and}$$

$$q \stackrel{\mathrm{def}}{=} \tau(x \cdot \mathrm{d}_{01} - \mathrm{d}_{12}) \leq x \rightarrow \tau(x \cdot \mathrm{d}_{01} - \mathrm{d}_{12}) = x \cdot \mathrm{d}_{01} - \mathrm{d}_{12}.$$

We claim that $\mathbf{F}NCA_\alpha \models q$ but $NCA_\alpha \not\models q$. Suppose that $\underline{C} \in NCA_\alpha$, $\underline{C} \not\models q$. We show that $\underline{C}$ is infinite. This will prove $\mathbf{F}NCA_\alpha \models q$. We may suppose that $\underline{C} \subseteq \underline{CmA}$ for some $\underline{A} \in NAt_\alpha$. Then $\underline{C} \not\models q$ means that there is an $H \subseteq E^{\underline{A}}_{01} \setminus E^{\underline{A}}_{12}$, $H \in C$ such that $\tau^{\underline{C}} H \subset H$. Since $\underline{A} \in NAt_\alpha$, we have

$$t^{\underline{A}}_{02} : E^{\underline{A}}_{01} \setminus E^{\underline{A}}_{12} \rightarrowtail\!\!\!\!\twoheadrightarrow E^{\underline{A}}_{02} \setminus E^{\underline{A}}_{12}$$

$$t^{\underline{A}}_{21} : E^{\underline{A}}_{02} \setminus E^{\underline{A}}_{12} \rightarrowtail\!\!\!\!\twoheadrightarrow E^{\underline{A}}_{12} \setminus E^{\underline{A}}_{01}, \text{ and}$$

$$t^{\underline{A}}_{10} : E^{\underline{A}}_{12} \setminus E^{\underline{A}}_{01} \rightarrowtail\!\!\!\!\twoheadrightarrow E^{\underline{A}}_{01} \setminus E^{\underline{A}}_{12}.$$

Let $f \stackrel{\mathrm{def}}{=} t^{\underline{A}}_{10} \circ t^{\underline{A}}_{21} \circ t^{\underline{A}}_{02}$. Then $f : E^{\underline{A}}_{01} \setminus E^{\underline{A}}_{12} \rightarrowtail\!\!\!\!\twoheadrightarrow E^{\underline{A}}_{01} \setminus E^{\underline{A}}_{12}$. Let $G \subseteq E^{\underline{A}}_{01} \setminus E^{\underline{A}}_{12}$, $G \in C$ be arbitrary. Then $\mathrm{t}^{1\underline{C}}_0 G = \mathrm{d}^{\underline{C}}_{10} \cap \mathrm{c}^{\underline{C}}_1 G = t^{\underline{A}\star}_{10} G$, and similarly for the other indices, so $\tau^{\underline{C}} G = f^\star G$. Since f is a bijection, we have $f^\star G \subset G \Rightarrow f^\star f^\star G \subset f^\star G$, thus $\tau H \subset H$ implies $H \supset \tau H \supset \tau\tau H \supset \ldots$, that is, H generates infinitely many different elements in $\underline{C}$. This proves $\mathbf{F}NCA_\alpha \models q$.

In order to show $NCA_\alpha \not\models q$ we present a specific $\underline{C} \in NCA_\alpha$ and $H \in C$ with $\tau^{\underline{C}} H < H$ but $H \leq \mathrm{d}^{\underline{C}}_{01} - \mathrm{d}^{\underline{C}}_{12}$ (see Fig. 2).

Let $\mathbb{Z}$ denote the set of integers, and let $e_0 \stackrel{\mathrm{def}}{=} \mathrm{Id} \restriction {}^2\{0,1\} \cup (\alpha \setminus 2)$. Let E denote those equivalence-relations over α in which exactly two different elements are equivalent, i.e.

$$E \stackrel{\mathrm{def}}{=} \{e \in \mathrm{Eqrel}(\alpha) : |e \setminus \mathrm{Id}| = 2\},$$

$$E' \stackrel{\mathrm{def}}{=} \mathrm{Eqrel}(\alpha) \setminus (E \cup \{\mathrm{Id}_\alpha\}),$$

$$A \stackrel{\mathrm{def}}{=} E \times \mathbb{Z} \cup \mathbb{E}' \times \{\not\vdash\},$$

$$E_{ij} \stackrel{\mathrm{def}}{=} \{(e, n) \in A : (i, j) \in e\},$$

$T_i \stackrel{\mathrm{def}}{=} \{\langle (e,n), (\bar{e}, m)\rangle \in {}^2A : e \restriction (\alpha \setminus \{i\}) = \bar{e} \restriction (\alpha \setminus \{i\}), [e \neq \bar{e} \text{ and } \{e, \bar{e}\} \subseteq E] \Rightarrow n = m, [e = \bar{e} \text{ and } i \in \mathrm{Dom}(e \setminus \mathrm{Id})] \Rightarrow n = m\}$, if $i \neq 1$,

$T_1 \stackrel{\mathrm{def}}{=} \{\langle (e,n), (\bar{e}, m)\rangle \in {}^2A : e \restriction (\alpha \setminus \{i\}) = \bar{e} \restriction (\alpha \setminus \{i\}), [e \neq \bar{e} \text{ and } \{e, \bar{e}\} \subseteq E \setminus \{e_0\}] \Rightarrow n = m, [e = \bar{e} \text{ and } 1 \in \mathrm{Dom}(e \setminus \mathrm{Id})] \Rightarrow n = m, e = e_0 \neq \bar{e} \in E \Rightarrow n = m+1, \bar{e} = e_0 \neq e \in E \Rightarrow m = n+1\}$,

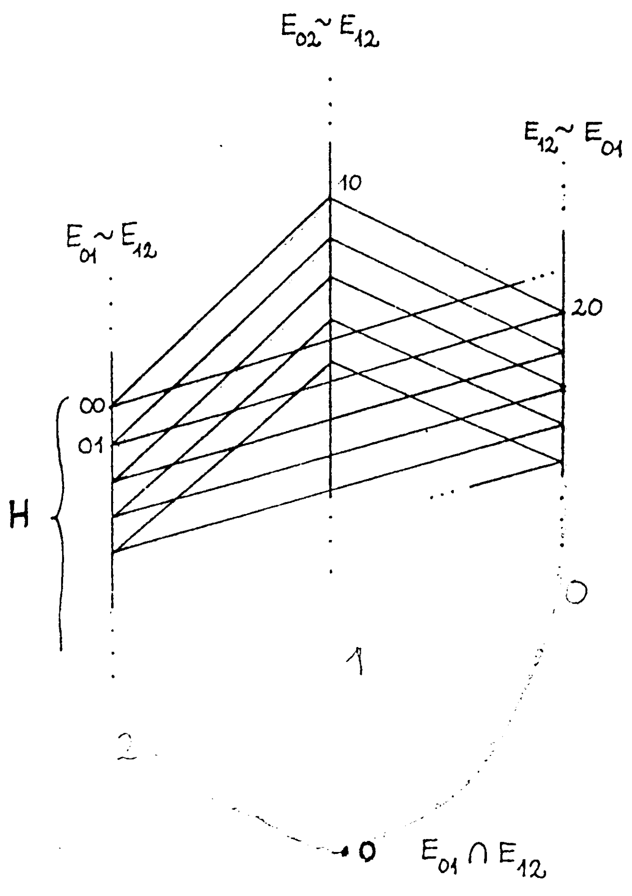


Figure 2.

$\underline{A} \stackrel{\text{def}}{=} \langle A, T_i, E_{ij}\rangle_{i,j\in\alpha}$.

It is easy to check that $\underline{A} \in NAt_\alpha$. Let $H \stackrel{\text{def}}{=} \{(e_0, n) : n \in \mathbb{Z}, \ltimes \geq \digamma\}$. Then it is easy to see that $H \subseteq E_{01}^{\underline{A}} \setminus E_{12}^{\underline{A}}$ and $\tau^{\underline{C}} H \subset H$, where $\underline{C} \stackrel{\text{def}}{=} \underline{CmA}$. ■(Theorem 5)

REMARK 3.4 (i) It is not possible to omit the condition $\alpha < \omega$ from Theorem 3.1(ii), since Eq$NCA_\alpha \neq$ Eq**F**NCA_α if $\alpha \geq \omega$: Let $\alpha \geq \omega$, $i, j \in \alpha$, $i \neq j$. Then $NCA_\alpha \not\models \mathrm{d}_{ij} = 1$, while $\mathbf{F}NCA_\alpha \models \mathrm{d}_{ij} = 1$ by the proof of [8] 1.3.12. Moreover, in Lemma 3.4 the condition ind$(\tau) \subset \beta$ cannot be omitted, since it was shown in [18] that there is an equation distinguishing NCA_α and NCA_β if $2 \leq \beta < \omega$, $\beta < \alpha$ (that is, we have $NCA_\beta \neq \mathbf{HSPRd}_\beta NCA_\alpha$ in this case). The same applies to the classes CA_α, see [8] 2.6.14(i).

In Theorem 3.1 we have seen that NCA_α is not strongly decidable if $\alpha \geq 3$. We do not know whether the word-problem for NCA_α is solvable for $\alpha \geq 3$. Below we show that for $\alpha \leq 2$, NCA_α is strongly decidable. Furthermore, ignoring the trivial cases, if $\alpha < \omega$, then omitting any nontrivial axiom-scheme beside C_4 gives a strongly decidable variety. (Here C_1, C_5 and C_6 are called trivial, because no variable-symbols occur in them.) For $i < 8$, let

$$NCA_\alpha^{-i} \stackrel{\text{def}}{=} \{\underline{A} \in CTA_\alpha : (\forall j \in 8 \setminus \{4, i\})\underline{A} \models C_j^\alpha\}.$$

Below we will show that if $0 < i < 8$, then

$$NCA_\alpha^{-i} \text{ is strongly decidable } \Longleftrightarrow i \in \{2, 3, 7\}.$$

It is plausible that NCA_α^{-0} is also strongly decidable. Instead of detailed proofs, we only give the constructions and the corresponding propositions, which can be easily checked. (We should perhaps note that it is usually easier to check C_3', C_3'' and C_3''' then C_3.)

Let $\underline{A} \in CTA_\alpha$, $\{\mathrm{d}_{ij} : i, j \in \alpha\} \subseteq X \subseteq Y \subseteq A$, $\alpha < \omega$, and let $\underline{B} \in CTA_\alpha$ be such that $B = \mathrm{Sg}^{\underline{BlA}} Y$, $\underline{BlB} \subseteq \underline{BlA}$, and $\mathrm{d}_{ij}^{\underline{B}} = \mathrm{d}_{ij}^{\underline{A}}$ if $i, j \in \alpha$.

(I) Suppose that $Y = X \cup \{\mathrm{s}_1^0 x, \mathrm{s}_0^1 x : x \in X\}$ and

$$\mathrm{c}_i^{\underline{B}} x = \prod\{y \in B : x \leq y = \mathrm{c}_i^{\underline{A}} y\}, \qquad \text{for all } i \in \alpha \text{ and } x \in B.$$

Then $\underline{A} \in NCA_\alpha^{-7} \Rightarrow (\underline{B} \in NCA_\alpha^{-7}$ and $\underline{A} \restriction X \subseteq \underline{B})$, and if $\alpha \leq 2$, then $\underline{A} \in NCA_\alpha \Rightarrow \underline{B} \in NCA_\alpha$. This proves the strong decidability of NCA_α for $\alpha \leq 2$, and that of NCA_α^{-7} for $\alpha < \omega$.

(II) Let $At \stackrel{\text{def}}{=} \mathrm{At}\underline{B}$ and suppose that $Y = X$ (where $\mathrm{At}\underline{B}$ denotes the set of atoms of $\underline{B}$).

1. Suppose that $\mathrm{c}_i^{\underline{B}} x = \sum\{a \in At : a \leq \mathrm{c}_i^{\underline{A}} x\}$, for all $i \in \alpha$ and $x \in B$. Then $\underline{A} \in NCA_\alpha^{-3} \Rightarrow (\underline{B} \in NCA_\alpha^{-3}$ and $\underline{A} \restriction X \subseteq \underline{B})$.

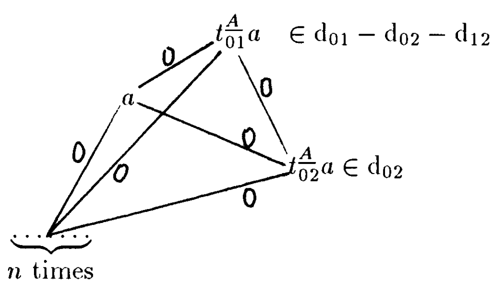
2. Suppose that $\underline{Bl A}$ is a Boolean set algebra over U. Let $At' \stackrel{\text{def}}{=} \mathrm{At}\{y \in B : y = c_i^{\underline{A}} y\}$. Let $r : At' \longrightarrow U$ be such that $\forall a \in At) r(a) \in A$ (a system of representatives). Suppose that $c_i^{\underline{B}} x = \sum\{a \in At' : r(a) \in x\}$ for all $i \in \alpha$, $x \in B$. Then $\underline{A} \in NCA_\alpha^{-2} \Rightarrow (\underline{B} \in NCA_\alpha^{-2}$ and $\underline{A} \restriction X \subseteq \underline{B})$.

The above cases prove the strong decidability of NCA_α^{-i} for $i \in \{2, 3, 7\}$.

Now let $i \in \{1, 5, 6\}$, and $q' \stackrel{\text{def}}{=} \bigwedge C_i^\alpha \to q$, where q is the quasi-equation of Theorem 3.1(iii). Then q' is easily seen to be equivalent to a quasi-equation, and (by Theorem 3.1(iii)) $\mathbf{F}NCA_\alpha^{-i} \models q'$ while $NCA_\alpha^{-i} \not\models q'$. Thus NCA_α^{-i} is not strongly decidable.

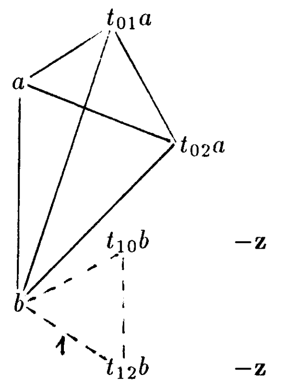
(ii) τ-trees were intentionaly called "trees": the present method is essentially the method of tree-proof widely used in logic for giving complete calculi (for instance the so called sequence-calculi are based on it). Below we present some simple examples in order to shed some light on the essence of the tree-proof method used in the proof of Theorem 3.1. At the same time we will show why the method in its present form cannot be used to decide $\mathrm{Eq}Crs_\alpha$.

Let $\tau = c_0(y - c_1 z)$, and suppose we want to decide whether $NCA_\alpha \models \tau = 0$. Let $\underline{A} \in NAt_\alpha$ and $a \in \tau^{\underline{C}}[k]$ for some $k : X \longrightarrow \mathrm{Sb}A$, where $\underline{C} = \underline{CmA}$. Then a has a 0-neighbour s (that is, $aT_0^{\underline{A}} s$) such that $s \in (y - c_1 z)^{\underline{C}}[k]$. Suppose that $E^{\underline{A}}(a) = \mathrm{Id} \restriction 3$ (i.e. $a \in -d_{01}^{\underline{C}} - d_{02}^{\underline{C}} - d_{12}^{\underline{C}}$). Then a may have four types of 0-neighbours in $\underline{A}$: $t_{00}^{\underline{A}} a = a$, $t_{01}^{\underline{A}} a$, $t_{02}^{\underline{A}} a$ (these are the ones that *must* be present), and an arbitrary number of 0-neighbours s with $E^{\underline{A}}(s) = \mathrm{Id} \restriction 3$. This is illustrated in the picture below:



Thus we have four cases to examine. Suppose that $b \in (y - c_1 z)^{\underline{C}}[k]$, where b is an "optional" neighbour of a, that is, $E^{\underline{A}}(b) = \mathrm{Id} \restriction 3$. Then $b \in k(y)$ and $b \notin (c_1 z)^{\underline{C}}[k]$. The latter means that no 1-neighbour of b is in $k(z)$. Now b has three different "obligatory" 1-neighbours: $t_{10}^{\underline{A}} b$, $t_{11}^{\underline{A}} b = b$ and $t_{12}^{\underline{A}} b$. Of these, $t_{10}^{\underline{A}} b$ and $t_{01}^{\underline{A}} a$ may coincide in principle but it will be useful to suppose that they do not. In picture:

$c_0(y - c_1 z)$



$-z, -c_1 z, y, y - c_1 z$

The picture "ended" with no "contradiction" (that is, there is no w such that both w and $-w$ are stipulated for some element). Thus our procedure gives $NCA_3 \not\models \tau = 0$. Executing the above procedure for $\sigma = c_0(y - c_1 y)$ in any possible way gives a contradiction, in accordance with $NCA_3 \models \sigma = 0$.

Now we illustrate the above procedure (explained in terms of pictures) in a real "tree-proof" form. We do not comment on this drawing since we hope it speaks for itself.

$$a \in c_0(y - c_1 z), \qquad a \in d_{01} - d_{02} - d_{12}$$

$$t_{00}a \in y - c_1 z \qquad t_{01}a \in y - c_1 z \qquad t_{02}a \in y - c_1 z \qquad t_{0\tau}a \in y - c_1 z$$

$$\vdots \qquad \vdots \qquad \vdots \qquad t_{0\tau}a \in y, \quad t_{0\tau}a \notin c_1 z$$

$$t_{0\tau}a \in y, \quad t_{10}t_{0\tau}a \notin z, \qquad t_{0\tau}a \notin z, \quad t_{12}t_{0\tau}a \notin z$$

We note that the τ-tree in Definition 3.2 corresponds to a "successful branch" of this "real tree", that is, it corresponds to a "proof".

It is natural to ask whether the tree-method in the proof of Theorem 3.1 can be used, with obvious modifications (using $\underline{At}(V)$'s instead of atomstructures), to decide Crs_α. For example we want to decide whether $Crs_3 \models c_0(y - c_1 z)$: Below we write abc for $\langle a, b, c\rangle$, etc.

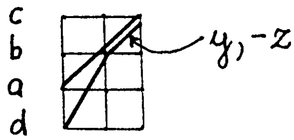
$$abc \in c_0 \overbrace{(y - c_1 z)}^{\sigma}, \qquad |\{a,b,c\}| = 3$$

$$bbc \in \sigma, \quad abc \in \sigma \quad cbc \in \sigma \quad dbc \in y - c_1 z, \quad |\{a,b,c\}| = 3$$

$$\vdots \qquad \vdots \qquad \vdots \qquad dbc \in y, \quad dbc \notin c_1 z$$

$$dbc \in y, \quad dbc \notin z$$

In the final step we made use of the fact that we are working in Crs_3, whence the only "obligatory" 1-neighbour of dbc is itself. Indeed, let $|\{a,b,c,d\}| = 4$, $V = \{abc, dbc\}$, $k(y) = \{dbc\}$ and $k(z) = 0$. Then $\underline{Sb}V \not\models \tau = 0[k]$, since $abc \in \tau^{\underline{Sb}V}[k]$. So $Crs_3 \not\models \tau = 0$.



This may seem to be a good procedure, but as the following example shows, it does not always terminate after a finite number of steps.

Let $\sigma \overset{\text{def}}{=} y - d_{01} \cdot c_0(d_{01} \cdot c_1(d_{12} - y))$ and $\tau \overset{\text{def}}{=} -d_{01} \cdot d_{12} - y - c_0 - c_1\sigma$.

$$abb \in \tau, \quad -d_{01}, \quad d_{12}, \quad -y, \quad -c_0 - c_1\sigma, \quad c_1\sigma$$

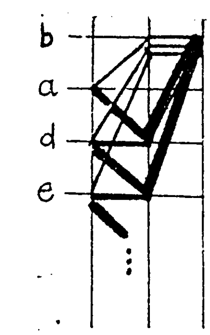
$$adb \in y, \quad -d_{01}, \quad c_0(d_{01} \cdot c_1(d_{12} - y))$$

$$ddb \in c_1(d_{12} - y)$$

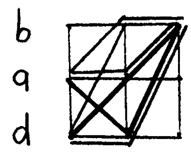
$$dbb \in -y, \quad c_1\sigma$$

$$deb \in y, \quad -d_{01}, \quad c_0(d_{01} \cdot c_1(d_{12} - y))$$

$$\vdots$$



It is easy to construct a Crs_3-unit V in which this "backward search" goes on indefinitely. For example, let $V \overset{\text{def}}{=} \bigcup\{{}^3\{0,n\} \cup {}^3\{n, n+1\} \cup \{\langle n, n+1, 0\rangle\} : n \in \omega\}$. Then $\underline{Sb}V \not\models \tau = 0$, but $\underline{Sb}W \models \tau = 0$ for all finite $W \subseteq V$. This τ cannot, however, be used to distinguish Crs_α's with finite and infinite units: There is a finite Crs_α-unit W with $\underline{Sb}W \not\models \tau = 0$, for example the one given below:



$$W = \{abb, aab, dbb, ddb, adb, dab\}$$
$$k(y) = \{dab, adb\}.$$

Indeed, the above method can be modified so as to yield this "solution", by trying as neighbours all the elements already occurring in the tree, instead of always introducing new elements:

$$
\begin{array}{lcr}
 & dbb \in -y, \quad c_1\sigma & \\
dab \in \sigma & & deb \in \sigma \\
dab \in y, \quad -d_{01}, \quad c_0(d_{01} \cdot c_1(d_{12} - y)) & & \vdots \\
aab \in c_1(d_{12} - y) & & \\
abb \in -y & &
\end{array}
$$

We do not know whether this tree-method is suitable for deciding the validity of equations in Crs_α. If it were, then $Crs_3 \not\models \tau = 0$ would imply the existence of a finite Crs_α-unit V with $\underline{Sb}V \not\models \tau = 0$. But we do not even know whether $\mathrm{Eq}Crs_3 = \mathrm{EqF}Crs_3$.

Thus we will use a different method for deciding[3] $\mathrm{Eq}Crs_\alpha$. (The new method can, however, be conceived of as constructing the model from the infinite tree. But then we have to be able to decide on the basis of a finite amount of information, whether there will eventually be a contradiction.) We note that almost all known decision-procedures are based on the construction of some finite model. ■

Before turning to our second main theorem (decidability of $\mathrm{Eq}Crs_\alpha$), let us prove a logical corollary of Theorem 3.1 (Corollary 3.1 below, cf. Theorem 1.1 in the Introduction). Roughly speaking, it says that it is the permutability of quantifiers that gives first-order logic its power, and so permutability is an essential feature of first order logic. More exactly: It is not hard to show that replacing ((4)) with four of its particular cases, viz.

((4a)) $\forall \mathbf{v}_i \forall \mathbf{v}_j \varphi \rightarrow \forall \mathbf{v}_j \forall \mathbf{v}_i \varphi$

((4b)) $\forall \mathbf{v}_k \varphi \rightarrow \forall \mathbf{v}_k \forall \mathbf{v}_k \varphi$

((4c)) $\exists \mathbf{v}_k \varphi \rightarrow \forall \mathbf{v}_k \exists \mathbf{v}_k \varphi$

[3]It is quite surprising that $\mathrm{Eq}Crs_\alpha$ is decidable at all – just the opposite was expected in light of other theorems on Crs_α.

((4d)) $R(\bar{x}) \longrightarrow \forall \mathbf{v}_k R(\bar{x})$ provided $\mathbf{v}_k \notin \mathrm{Rng}\bar{x}$ and $R(\bar{x})$ is an atomic formula,

leaves the proof-system (i.e. the set of formulas provable in) $\vdash_\alpha$ unchanged. [((4a)) is a special case since it comes from $\forall \mathbf{v}_i \forall \mathbf{v}_k \varphi \longrightarrow \forall \mathbf{v}_k \forall \mathbf{v}_i \forall \mathbf{v}_k \varphi$ by a straightforward application of ((3a)), ((2)), and this is an instance of ((4)).] It will be shown below that leaving ((4a)) out of $\vdash_\alpha$ yields a substantially weaker proof-system.

COROLLARY 3.1 *The proof-system obtained by replacing ((4)) with ((4b)) ... ((4d)) is substantially weaker than $\vdash_\alpha$. Namely, the set of formulas derivable from ((1))... ((3)), ((4b))... ((4d)), ((5))... ((9)) by (MP) and (G) is decidable.*

Proof: Let $\vdash'$ denote the weakened proof-system (i.e. $\vdash'$ is $\vdash_\alpha$ without ((4a))). Let $\Lambda = \langle \alpha, t \rangle$ be a first-order language, where $t : \underline{R} \longrightarrow \omega$. Let $\equiv' \stackrel{\text{def}}{=} \{(\varphi, \psi) \in {}^2\mathrm{Fm}^\Lambda : \vdash' \varphi \leftrightarrow \psi\}$ and let $\underline{F} \stackrel{\text{def}}{=} \underline{Fm}^\Lambda / \equiv'$. First we show that $\underline{F}$ is the free NCA_α dimension-restricted by t, that is, $\underline{F} \cong \underline{\mathcal{F}}^{(t)}_{\underline{R}} NCA_\alpha$, and then we show that by "adjusting" the concept of a τ-tree to t, we obtain a decision procedure for the congruence $\mathrm{Cr}^{(t)}_{\underline{R}} NCA_\alpha$ (as a set of pairs). In order to prove $\underline{F} \cong \underline{\mathcal{F}}^{(t)}_{\underline{R}} NCA_\alpha$, we have to show that

(a) $\underline{F} \in NCA_\alpha$

(b) In $\underline{\mathcal{F}}^{(t)}_{\underline{R}} NCA_\alpha$, ((1))...((9))\((4a)) are in the same class as *True*

(see the proof of [8] 4.3.25 for details).
On the proof of (a): Here one has to repeat the proof of [8] 4.3.22 in a slightly different setting (that is, we cannot use ((4)), and don't have to prove C_4). ((4)) is used four times in the proof of [8] 4.3.22 (not counting its uses in proving C_4). For the first three uses (on p.158_{13}, in (e) on p.158, and in (h) on p.159 in Part II) one can substitute ((4b)), ((4c)) and ((4d)), respectively. As for the last one on p.159_{18}, note that

$$\begin{array}{ll} \mathrm{c}_\lambda \mathrm{c}_\kappa (\mathrm{d}_{\lambda\kappa} \cdot \mathrm{d}_{\kappa\mu}) & \\ = \mathrm{c}_\lambda \mathrm{c}_\kappa (\mathrm{d}_{\lambda\mu} \cdot \mathrm{d}_{\kappa\mu}) & \text{by ((7)) (and ((1)),((5)))} \\ = \mathrm{c}_\lambda (\mathrm{c}_\kappa \mathrm{d}_{\lambda\mu} \cdot \mathrm{c}_\kappa \mathrm{d}_{\kappa\mu}) & \text{by (h)},C_3 \\ = 1, & \text{by ((6)),(h)} \end{array}$$

where (h) is the statement from the proof of [8] 4.3.22.
On the proof of (b): This is proved (without using C_4) in [8] 4.3.25 for ((1)),((2)) and ((8)). The other claims are direct consequences of axioms of NCA_α.

Let $\tau \in \mathrm{Tm}_{\underline{R}}(\mathrm{cyl}_\alpha)$. We may suppose $\alpha < \omega$. P is said to be a t, τ-*tree* if

(i) P is a τ-tree, and

(ii) for all $R \in \underline{R}, i \in \alpha \setminus tR, n \in \alpha \cup \mathrm{Tm}$ and $w \in \mathrm{Tm}(t_\alpha)$, if $\{w, t_{in}w\} \subseteq \mathrm{Dom}P$, then $[(w, R) \in P \leftrightarrow (t_{in}w, R) \in P]$.

A slight modification of the proof of Theorem 3.1 shows that

$$(\tau, 0) \in \mathrm{Cr}^{(t)}_{\underline{R}} NCA_\alpha \Leftrightarrow \text{ there is no } t, \tau\text{-tree},$$

and this yields a decision procedure for $\mathrm{Cr}^{(t)}_{\underline{R}} NCA_\alpha$. Summing up, the decision procedure for the proof-system $\vdash'$ is the following: Let $\varphi \in \mathrm{Fm}^\Lambda$ and let $\tau\mu(\varphi)$ be the term in $\mathrm{Tm}_{\underline{R}}(\mathrm{cyl}_\alpha)$ corresponding to φ. Then $[\vdash' \varphi \Leftrightarrow$ there is a $t, \tau\mu(\varphi)$-tree], and the latter is decidable. ■(Corollary 3.1)

4 Deciding the equational theory of cylindric-relativized set algebras

We begin the investigation of the class Crs_α and the model-theoretic significance of the permutability of quantifiers. Let

$$\begin{aligned} WCA_\alpha \stackrel{\text{def}}{=} \; & \{\underline{A} \in NCA_\alpha^{-6} : \underline{A} \models \mathrm{d}_{ik} \cdot \mathrm{d}_{kj} \leq \mathrm{d}_{ij} = \mathrm{d}_{ji} = \mathrm{c}_k \mathrm{d}_{ji} \\ & \text{if } i, j, k \in \alpha, k \notin \{i, j\}\}. \end{aligned}$$

That is, WCA_α is the class of those algebras from CTA_α which satisfy a weakened version of C_6 and all CA_α axioms except C_4 and C_6. Then $Crs_\alpha \subseteq WCA_\alpha$ and WCA_α is decidable but not strongly (this can be seen from the proof of Theorem 3.1, since $WCA_\alpha = \mathbf{ISCmp}NAt_\alpha$). Crs_α could be called the representable part of WCA_α. If $\alpha \leq 2$, then $\mathbf{I}Crs_a = WCA_\alpha$ by [8] 5.5.5 (this is a theorem of Henkin and Resek). By Theorem 4.1 below, for $\alpha \geq 3$, $\mathbf{I}Crs_\alpha$ is axiomatizable by identities, but not by finitely many ones (or not by finitely many schemes when $\alpha \geq \omega$), so $\mathbf{I}Crs_\alpha \subset WCA_\alpha$, since the latter is defined by finitely many schemes. We do not define the concept of a scheme (or rather, a scheme of equation) since we do not need it later. We note however, that the concept of a scheme is a quite natural one, e.g. "$\mathrm{c}_i \mathrm{d}_{ij} = 1$ if $i, j \in \alpha$" is a scheme. For α infinite, schemes are more important than the equations themselves, since in this case, the similarity type being infinite, hardly anything can be defined by finitely many equations, while usually one can define quite a lot of things by finitely many schemes. The definition of a scheme can be found for example in [8] 4.1.4; for more on this notion and its importance, see [2], [3].

We cite Theorem 4.1 below (without proof) as a source of motivation. The proof of (i) and (iii) was published in [15], and that of (ii) and (iii) in [16]. All these proofs are cited in the monograph [8], see 5.5.10, 5.5.12, 5.5.13 and 5.5.16.

THEOREM 4.1 *Let α be an arbitrary set.*

(i) $\mathbf{I}Crs_\alpha$ *is a variety, i.e. it is axiomatizable with identities.*

(ii) $\mathbf{I}Crs_\alpha$ *is not finitely axiomatizable.*

(iii) $\mathbf{I}Crs_\alpha$ *is not axiomatizable by finitely many schemes, but it is axiomatizable with countably many schemes.*

■

Our main concern in this section is showing that $\mathrm{Eq}Crs_\alpha$ is decidable for all (decidable) α.

Before beginning the proof we define two subclasses of Crs_α, since we are going to prove the theorem for these classes, too.

DEFINITION 4.1 Let α be an arbitrary set.

$$D_\alpha \stackrel{\mathrm{def}}{=} \{\underline{A} \in Crs_\alpha : (\forall s \in 1^{\underline{A}})(\forall i, j \in \alpha) s(i/s_j) \in 1^{\underline{A}}\},$$

$$G_\alpha \stackrel{\mathrm{def}}{=} \{\underline{A} \in Crs_\alpha : (\forall s \in 1^{\underline{A}})^\alpha(\mathrm{Rng}s) \subseteq 1^{\underline{A}}\}.$$

Let $K \subseteq Crs_\alpha$. Then V is a K-unit if $\underline{Sb}V \in K$. V is straightenable if V is a D_α-unit. ■

Obviously $Gs_\alpha \subseteq G_\alpha \subseteq D_\alpha \subseteq Crs_\alpha$. The classes $\mathbf{I}Gs_\alpha$, $\mathbf{I}G_\alpha$, $\mathbf{I}D_\alpha$ and $\mathbf{I}Crs_\alpha$ are all varieties, and they are different if $\alpha \geq 2$: It is well-known that $\mathbf{I}Gs_\alpha$ is a variety (see [8] 3.1.108); a modification of the proof of [9] I.7.27 (p.114) shows that $\mathbf{I}G_\alpha$ is a variety if α is finite. For infinite α we do not know whether G_α is a variety. $\mathbf{I}Crs_\alpha$ is a variety by Theorem 4.1. It follows that $\mathbf{I}D_\alpha$ is a variety, too, since it is easy to see that $D_\alpha = \{\underline{A} \in Crs_\alpha : \underline{A} \models \mathsf{c}_i\mathsf{d}_{ij} = 1 \text{ for all } i, j \in \alpha\} = \{\underline{A} \in Crs_\alpha : \underline{A} \models C_6\}$ and thus $Crs_\alpha \cap CA_\alpha = \{\underline{A} \in D_\alpha : \underline{A} \models C_4\}$. We will see later that for $\alpha < \omega$, $\mathrm{Eq}D_\alpha$ is decidable, while $\mathrm{Eq}(Crs_\alpha \cap CA_\alpha)$ is known to be undecidable. Thus omitting C_4 yields decidability again. We note that $G_\alpha \not\subseteq CA_\alpha$, that is, $G_\alpha \not\models C_4$. So the identity C_4 distinguishes Gs_α and G_α, while the identity C_6 distinguishes D_α and Crs_α. (In the proof of Theorem 4.3 we will give an equation distinguishing G_α and D_α.) Let V be a G_α-unit. Then it is not hard to see that $V = \bigcup\{{}^\alpha U_i : i \in I\}$ for some family of sets $\langle U_i : i \in I\rangle$. Since the definition of a Gs_α-unit is the same but for one additional constraint, viz. that the U_i's must be pairwise disjoint, the elements of G_α will sometimes be called "non-disjoint Gs_α's", and V will be called a "non-disjoint Gs_α-unit". Below we prove the decidability of $\mathrm{Eq}G_\alpha$, while $\mathrm{Eq}Gs_\alpha$ is known to be undecidable. Thus the apparently innocent disjointness condition in the definition of Gs_α-unit turns out to be essential. We note that $G_\alpha \subseteq D_\alpha \subseteq NCA_\alpha$.

THEOREM 4.2 *1. $\mathrm{Eq}G_\alpha$ and $\mathrm{Eq}Crs_\alpha$ are decidable for all $\alpha \leq \omega$*

2. $\mathrm{Eq}D_\alpha$ is decidable, provided $\alpha < \omega$.

3. Let $\alpha < \omega$ and suppose that $K \subseteq Crs_\alpha$ satisfies the following conditions:

(a) *The union of K-units is a K-unit, i.e.* $(\forall V \in W)\underline{Sb}V \in K \Rightarrow \underline{Sb}(\bigcup W) \in K$.

(b) *The restriction of a K-unit is a K-unit, i.e.* $\underline{Sb}V \in K \Rightarrow \underline{Sb}(V \cap {}^{\alpha}H) \in K$.

(c) *The "base-isomorphic" image of a K-unit is a K-unit, i.e.*

$$\underline{Sb}V \in K \text{ and } f : \text{base}(V) \rightarrowtail U \Rightarrow \underline{Sb}\{f \circ s : s \in V\} \in K.$$

(d) $\underline{A} \in K \Rightarrow \underline{Sb}1^{\underline{A}} \in K$.

Then EqK *is decidable.*

Proof: We use some of the notations (like Tm, Subterm, ind) introduced in the proof of Theorem 3.1. First we prove (iii).

Let $\alpha < \omega$ and suppose that $K \subseteq Crs_\alpha$ satisfies conditions (a)...(d) of the theorem. It suffices to give a decision procedure for the set $\{\tau \in \text{Tm} : K \models \tau = 1\}$, since

$$K \models \delta = \sigma \iff K \models -(\delta \oplus \sigma) = 1,$$

where $\oplus$ denotes Boolean symmetric difference[4]. We may suppose that $K = \mathbf{S}K$, for EqK = Eq$\mathbf{S}K$ and if K satisfies conditions (a)...(d) then so does $\mathbf{S}K$; we may further assume $\alpha \geq 2$, since $\alpha \leq 1$, $K = \mathbf{S}K$ and (a)...(d) imply $K = Crs_\alpha = Gs_\alpha$, and Gs_α is decidable for $\alpha \leq 1$, see [8] §4.2.

DEFINITION 4.2 (i) Let E be a Crs_α-unit. Then $\delta(E)$ denotes the smallest D_α-unit containing E. (There is such a D_α-unit since the intersection of D_α-units is a D_α-unit.)

(ii) Let $\tau \in \text{Tm}$. Then (E, P) is said to be a τ, K-<u>mosaic</u> on U (or simply a mosaic) if conditions 1, 2 below hold:

1. E is a K-unit, and $U = \text{base}(E)$,
2. $P : \text{Subterm}(\tau) \longrightarrow \text{Sb}\delta(E)$ such that
   (a) $P(\mathrm{d}_{ij}) = \mathrm{D}_{ij}^{[E]} = \{s \in E : s(i) = s(j)\}$ if $\mathrm{d}_{ij} \in \text{Subterm}(\tau)$,
   (b) $P(\sigma \cdot \delta) = P(\sigma) \cap P(\delta) \cap E$ if $\sigma \cdot \delta \in \text{Subterm}(\tau)$,
   (c) $P(-\sigma) = E \setminus P(\sigma)$ if $-\sigma \in \text{Subterm}(\tau)$,
   (d) $P(\sigma) \cap E \subseteq P(\mathrm{c}_i\sigma) = \mathrm{C}_i^{[\delta(E)]}P(\mathrm{c}_i\sigma)$ if $\mathrm{c}_i\sigma \in \text{Subterm}(\tau)$.

■

[4] In this proof it is not important, but convenient, that we use identities of the form $\tau = 1$. If we wanted to decide $\tau = \sigma$ directly, then we would have to replace Subterm(τ) by Subterm(τ) $\cup$ Subterm(σ) in subsequent definitions, and write "$(E,P) \models \tau = \sigma \Leftrightarrow P(\tau) \cap E = P(\sigma) \cap E$" instead of "$(E,P) \models \tau = 1 \Leftrightarrow P(\tau) \cap E = E$".

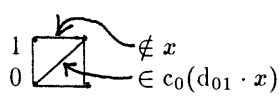
REMARK 4.1 (i) We show that a mosaic is nothing but a finite piece "cut out" from an algebra-valuation pair. Let (E, P) be an algebra-valuation pair satisfying the stronger condition (in items (i) and (ii) of this Remark we always assume $\delta(E) = E$ for simplicity's sake)

$$(\mathrm{d}') \qquad \mathrm{P}(\mathrm{c_i}\sigma) = \mathrm{C_i^{[E]}P}(\sigma) \text{ if } \mathrm{c_i}\sigma \in \mathrm{Subterm}(\tau).$$

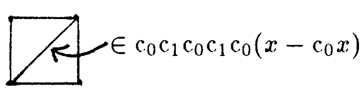
Then it is easy to see that $P(\sigma) = \sigma^{\underline{Sb}E}[P \upharpoonright X]$ for all $\sigma \in \mathrm{Subterm}(\tau)$. Conversely, if $\underline{A} \in K$, and $k : X \longrightarrow A$, then $(1^{\underline{A}}, \langle \sigma^{\underline{A}}[k] : \sigma \in \mathrm{Subterm}(\tau)\rangle)$ is a mosaic satisfying the stronger condition (d'). Thus there is a one to one correspondence between mosaics satisfying the stronger condition (d') and algebra-valuation pairs. If (E, P) is a mosaic satisfying the stronger condition (d') then we say that (E, P) is a τ, K-algebra-valuation pair, briefly τ, K-Avp. Now let (E, P) be a τ, K-Avp and let $E' \subseteq E$ be arbitrary (D_α-unit). Then it is not hard to see that

$$(E', \langle P(\sigma) \cap E' : \sigma \in \mathrm{Subterm}(\tau)\rangle) \text{ is a } \tau, K\text{-mosaic.}$$

So far the situation is analogous to the one encountered when we were considering $NAt_\alpha \setminus pNAt_\alpha$. But here the analogy breaks down since $pNAt_\alpha$ is a complete description of $\{\underline{A} \upharpoonright W : \underline{A} \in NAt_\alpha\}$ (that is, all $pNAt_\alpha$ can be obtained from an NAt_α by "cutting out"), while there are mosaics which cannot be obtained by "cutting out" from a τ, K-Avp, for example the mosaic (E, P) shown below:



Here $\alpha = 2$, $E = {}^22$, $\tau = \mathrm{c_0}(\mathrm{d_{01}} \cdot x)$ and $P = \{(x, 0), (\mathrm{d_{01}}, \mathrm{D_{01}^{[^22]}}), (\mathrm{d_{01}} \cdot \mathrm{x}, 0), (\tau, \{(0, 1)\})\}$. But there are mosaics where the "discrepancy" only comes to light at a later stage, as in the following:



Thus in the present proof not mosaics, but good sets of mosaics will play a crucial role: it will be true for the notion of complete set of mosaics, to be defined below, that every complete set of mosaics is the set of all "cut out" mosaics coming from a τ, K-Avp. We note that the τ-tree in the previous proof is also part of some algebra-valuation pair.

(ii) If (E, P) is a mosaic, then the only way P may fall short of being a "real meaning-function" is to have the "shortcoming"

$$s \in P(c_i\sigma) \text{ and } (\forall u)s(i/u) \notin P(\sigma).$$

These "defects" will be repaired step by step (by adjoining a new sequence $s(i/u)$ to E, which will required to be in $P(\sigma)$, cf. the definition of $\mathcal{M}$-continuable below). The advantage of mosaics over algebra-valuation pairs is their "smallness": while there are mosaics on $U \subseteq \alpha$, there may be no algebra-valuation pairs on any finite U.
(iii) In Remark 4.2 we will discuss the reason why should one "take into account the valuation" on $\delta(E)$, too (in the Crs_α case). ∎

DEFINITION 4.3 Let (E, P) and (E', P') be τ, K-mosaics on U and U', respectively.

(i) Let $W \subseteq U$. Then

$$(E, P) \restriction W \stackrel{\text{def}}{=} (E \cap {}^\alpha W, \langle P(\sigma) \cap {}^\alpha W : \sigma \in \text{Subterm}(\tau)\rangle).$$

(E', P') is said to be an extension of (E, P) (or $(E, P) \prec (E', P')$), if $(E, P) = (E', P') \restriction U$.

(ii) Isomorphism of τ, K-mosaics is defined in the usual way: f is an isomorphism between (E, P) and (E', P') (or $f : (E, P) \rightarrowtail\!\!\!\twoheadrightarrow (E', P')$), if $f : U \rightarrowtail\!\!\!\twoheadrightarrow U'$, $E' = \{f \circ s : s \in E\}$, and $P' = \langle\{f \circ s : s \in P(\sigma)\} : \sigma \in \text{Subterm}(\tau)\rangle$. If $\mathcal{M}$ is a set of mosaics, then $\mathbf{I}\mathcal{M}$ denotes the class of mosaics isomorphic to elements of $\mathcal{M}$.

∎

DEFINITION 4.4 Let $\mathcal{M}$ be a set of mosaics and let (E, P) be a mosaic on U.

(i) (E, P) is $\mathcal{M}$-like if $(\forall s \in E)(E, P) \restriction H \in \mathbf{I}\mathcal{M}$, for some $H \supseteq \text{Rng}s$.

(ii) (E, P) is $\mathcal{M}$-continuable if for all $i \in \alpha$, $c_i\sigma \in \text{Subterm}(\tau)$ and $s \in P(c_i\sigma) \cap E$, there is an $\mathcal{M}$-like extension (E', P') of (E, P), in which $(\exists u)s(i/u) \in P'(\sigma) \cap E'$.

(iii) $\mathcal{M}$ is complete if all its elements are $\mathcal{M}$-continuable.

(iv) $\mathcal{M} \models \tau \stackrel{\text{def}}{\iff} (\forall M \in \mathcal{M})M \models \tau$, where $(E, P) \models \tau \stackrel{\text{def}}{\iff} P(\tau) \supseteq E$.

(v) $\Pi(\tau, K)$ (or simply Π) denotes the set of those mosaics (E, P) for which $E \subseteq {}^\alpha\alpha$.

∎

Claim 4.1 *$K \models \tau = 1 \iff [\mathcal{M} \models \tau$ for all complete $\mathcal{M} \subseteq \Pi(\tau, K)]$.*

Claim 4.2 *It is decidable whether "$\mathcal{M} \models \tau$ for all complete $\mathcal{M} \subseteq \Pi$" holds.*

Before giving the proofs of these claims, we collect the basic properties of mosaics in a Lemma.

DEFINITION 4.5 (i) Let $M = (E, P)$ and $M' = (E', P')$ be mosaics on U and U', respectively. Then M and M' are said to be compatible if $M \restriction (U \cap U') = M' \restriction (U \cap U')$.

(ii) Let $\mathcal{M}$ be a set of mosaics. Then $\bigcup \mathcal{M} \stackrel{\text{def}}{=} (E, P)$, where $E \stackrel{\text{def}}{=} \bigcup\{E' : (E', P') \in \mathcal{M}\}$, and $P(\sigma) = \bigcup\{P'(\sigma) : (E', P') \in \mathcal{M}\}$ for all $\sigma \in \text{Subterm}(\tau)$.

■

LEMMA 4.1 *Let $\mathcal{M}$ be a set of mosaics and (E, P) a mosaic on U.*

(i) *Let $W \subseteq U$, $f : U \rightarrowtail\!\!\twoheadrightarrow U'$, $E' \stackrel{\text{def}}{=} \{f \circ s : s \in E\}$ and $P' = \langle\{f \circ s : s \in P(\sigma)\} : \sigma \in \text{Subterm}(\tau)\rangle$. Then both (E', P') and $(E, P) \restriction W$ are mosaics, and both are $\mathcal{M}$-like ($\mathcal{M}$-continuable) provided (E, P) is $\mathcal{M}$-like ($\mathcal{M}$- continuable).*

(ii) *Let $\mathcal{P}$ be a set of pairwise compatible mosaics. Then $\bigcup \mathcal{P}$ is a mosaic, and $(E', P') \prec \bigcup \mathcal{P}$ for all $(E', P') \in \mathcal{P}$. Moreover, $\bigcup \mathcal{P}$ is $\mathcal{M}$-like provided all elements of $\mathcal{P}$ are $\mathcal{M}$-like.*

(iii) *If $\mathcal{M}$ is complete, then $[(E, P)$ is $\mathcal{M}$-like $\Rightarrow (E, P)$ is $\mathcal{M}$-continuable$]$.*

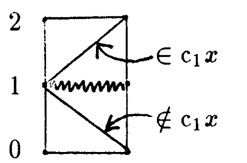
Proof: It is routine to check (i). As for the proof of (ii): Let $\bigcup \mathcal{P} = (E, P)$. Then E, being a union of K-units, is a K-unit, and $\delta E = \bigcup\{\delta E' : (E', P') \in \mathcal{P}\}$, since the union of D_α-units is a D_α-unit. Thus $P : \text{Subterm}(\tau) \longrightarrow \text{Sb}\delta(E)$. To show that $\bigcup \mathcal{P}$ is a mosaic, first we check condition (2d). Let $i \in \alpha$ and $\mathsf{c}_i\sigma \in \text{Subterm}(\tau)$. Let $s \in P(\sigma) \cap E$. Then there are $M' = (P', E') \in \mathcal{P}$ and $M'' = (P'', E'') \in \mathcal{P}$ with $s \in P'(\sigma)$ and $s \in E''$. Now $s \in P'(\sigma)$ gives $s \in \delta E'$ and thus $s \in {}^\alpha\text{base}(E')$ since $\text{base}(E') = \text{base}(\delta E')$. M' and M'' are compatible, so $s \in E'$ follows from $s \in E''$. Thus $s \in P'(\sigma) \cap E' \subseteq P'(\mathsf{c}_i\sigma) \subseteq P(\mathsf{c}_i\sigma)$. With this we have shown $P(\sigma) \cap E \subseteq P(\mathsf{c}_i\sigma)$. It remains to prove $\mathsf{C}_i^{[\delta(E)]}P(\mathsf{c}_i\sigma) \subseteq P(\mathsf{c}_i\sigma)$. Let $s \in \mathsf{C}_i^{[\delta(E)]}P(\mathsf{c}_i\sigma)$. Then $s \in \delta E'$ for some $M' = (E', P') \in \mathcal{P}$, and $z \stackrel{\text{def}}{=} s(i/u) \in P''(\mathsf{c}_i\sigma)$ for some u and $M'' = (E'', P'') \in \mathcal{P}$. Let $j \in \alpha \setminus \{i\}$ (recall that we have assumed $\alpha \geq 2$), and let $w \stackrel{\text{def}}{=} s(i/s_j)$. Then $w = z(i/z_j)$, thus $w \in \delta E''$, and so $w \in P''(\mathsf{c}_i\sigma)$ since $z \in P''(\mathsf{c}_i\sigma)$. But then $w \in P'(\mathsf{c}_i\sigma)$, since $\text{Rng}w \subseteq \text{Rng}z \cap \text{Rng}s$, and M'' and M' are compatible. Thus $s \in P'(\mathsf{c}_i\sigma) \subseteq P(\mathsf{c}_i\sigma)$ since $s \in \delta E'$. We have thus shown that $\bigcup \mathcal{P}$ satisfies condition (2d). The other conditions in (2) are easier to check: $P(\mathrm{d}_{ij}) = \bigcup\{P'(\mathrm{d}_{ij}) : (E', P') \in \mathcal{P}\} = \bigcup\{\mathrm{D}_{ij}^{[E']} : (E', P') \in \mathcal{P}\} = \mathrm{D}_{ij}^{[E]}$. Let $s \in P(\sigma \cdot \delta)$. Then $(\exists (E', P') \in \mathcal{P}) s \in P'(\sigma \cdot \delta) = P'(\sigma) \cap P'(\delta) \cap E' \subseteq P(\sigma) \cap P(\delta) \cap E$. Suppose that $s \in P(\sigma) \cap P(\delta) \cap E$. Then there are $M' = (E', P') \in \mathcal{P}$, $M'' = (E'', P'') \in \mathcal{P}$ and $M''' = (E''', P''') \in \mathcal{P}$ with $s \in P'(\sigma)$, $s \in P''(\delta)$ and $s \in E'''$. Since M', M'' and M''' are compatible, we have $s \in E'$, $s \in P'(\delta)$, and thus $s \in P'(\sigma) \cap P'(\delta) \cap E' \subseteq P'(\sigma \cdot \delta) \subseteq P(\sigma \cdot \delta)$. We have thus shown $P(\sigma \cdot \delta) = P(\sigma) \cap P(\delta) \cap E$. Now suppose that $s \in P(-\sigma)$, say $s \in P''(-\sigma) \subseteq E''$ for some $M'' = (E'', P'') \in \mathcal{P}$. Let $M' = (E', P') \in \mathcal{P}$ be arbitrary. If $s \in \delta(E')$, then $s \in P'(-\sigma)$ since M'' and M' are compatible, so $s \notin P'(\sigma)$. If $s \notin \delta(E')$, then clearly $s \notin P'(\sigma)$. So $s \notin \bigcup\{P'(\sigma) : (E', P') \in \mathcal{P}\} = P(\sigma)$. Suppose that $s \in E \setminus P(\sigma)$. Then $s \in E'$ and $s \notin P'(\sigma)$ for some $(E', P') \in \mathcal{P}$, whence $s \in E' \setminus P'(\sigma) = P'(-\sigma) = P(-\sigma)$. That is, $P(-\sigma) = E \setminus P(\sigma)$. With this we have checked that $\bigcup \mathcal{P}$ is a mosaic.

Let $M' = (E', P')$ be arbitrary. We show that $M' \prec \bigcup \mathcal{P}$. Let $W = \mathrm{base}(E')$ and $\sigma \in \mathrm{Subterm}(\tau)$. Then $E' \subseteq E \cap {}^{\alpha}W$ and $P'(\sigma) \subseteq P(\sigma) \cap {}^{\alpha}W$ as can be seen from the definition. Let $s \in E \cap {}^{\alpha}W$. Then $s \in E''$ for some $M'' = (E'', P'') \in \mathcal{P}$. We have $(s \in E'' \Rightarrow s \in E')$, since M' and M'' are compatible. So $E' = E \cap {}^{\alpha}W$. The proof of $P'(\sigma) = P(\sigma) \cap {}^{\alpha}W$ proceeds similarly: Let $s \in P(\sigma) \cap {}^{\alpha}W$. Then $s \in P''(\sigma)$ for some $M'' = (E'', P'') \in \mathcal{P}$, and then $s \in P'(\sigma)$, since M'' and M' are compatible. We have thus proved $M' \prec \bigcup \mathcal{P}$.

Suppose that all elements of $\mathcal{P}$ are $\mathcal{M}$-like, and let $e \in E$. Then $s \in E'$ for some $M' = (E', P') \in \mathcal{P}$. Then $\bigcup \mathcal{P} \upharpoonright \mathrm{Rng}s = M' \upharpoonright \mathrm{Rng}s \in \mathbf{I}\mathcal{M}$, since M' is $\mathcal{M}$-like. This finishes the proof of (ii).

To prove (iii), suppose that $M = (E, P)$ is $\mathcal{M}$-like. Let $i \in \alpha$, $\mathrm{c}_i\sigma \in \mathrm{Subterm}(\tau)$, $s \in P(\mathrm{c}_i\sigma) \cap E$, and $W \stackrel{\mathrm{def}}{=} \mathrm{Rng}s$. Then $M \upharpoonright W \in \mathbf{I}\mathcal{M}$, since M is $\mathcal{M}$-like, whence $M \upharpoonright W$ is $\mathcal{M}$-continuable by (i), since $\mathcal{M}$ is complete. Let $M' = (E', P')$ be an $\mathcal{M}$-like extension of $M \upharpoonright W$ such that $(\exists u) s(i/u) \in P'(\mathrm{c}_i\sigma) \cap E'$. We can assume $\mathrm{base}(E) \cap \mathrm{base}(E') = W$ by (i). But then M and M' are compatible, and thus $M'' = M \cup M'$ is a mosaic, and $M \prec M''$. Let $M'' = (E'', P'')$. Then obviously $(\exists u) s(i/u) \in P''(\mathrm{c}_i\sigma) \cap E''$. Moreover, M'' is $\mathcal{M}$-like, since both M and M' are $\mathcal{M}$-like. Thus M is $\mathcal{M}$-continuable. ■(Lemma 4.1)

REMARK 4.2 (i) The reason to require $P(\mathrm{c}_i\sigma) = C_i^{[\delta(E)]} P(\mathrm{c}_i\sigma)$ in Definition 4.2(ii)(2d) instead of just $P(\mathrm{c}_i\sigma) = C_i^{[E]} P(\mathrm{c}_i\sigma)$ in the Crs_α-case is that in this way the "contradiction" will show up on the intersection of the mosaics one attempts to match, cf. the proof of Lemma 4.1(ii). For example, let $\alpha = 2$, $\tau = \mathrm{c}_1 x$, $E = \{(1,2)\}$, $P = \{(x, 0), (\mathrm{c}_1 x, \{(1,2)\})\}$, $E' = \{(1,0)\}$ and $P' = \{(x, 0), (\mathrm{c}_1 x, 0)\}$, see the picture below.



(E, P)

(E', P')

Then it takes computing to find out that (E, P) and (E', P') cannot be matched. The present definition requires $P(c_1x) = \{(1, 2), (1, 1)\}$ for (E, P) to be a mosaic, and then clearly $(1, 1) \in P(c_1x)$, while $(1, 1) \notin P'(c_1x)$, so that P and P' differ with respect to the sequence $(1, 1)$.
(ii) The "key point" in the present proof is Lemmma 4.1(ii). It says that "continuability" is a local property, i.e. whether a sequence s can be "repaired" in a mosaic M depends only on $M \restriction \mathrm{Rng}s$ and nothing beyond it. (That is why a model decomposes into a set of mosaics, or more exactly, that is why the model can be restored from the set of mosaics derived from it.) ■

Proof of Claim 4.2: It is easy to check that Π is finite, and if $(E, P) \in \Pi$, then $E, \delta E, P$ and U are finite, too. Thus there are finitely many $\mathcal{M} \subseteq \Pi$. Since $P(\tau) \supseteq E$ is decidable for all $(E, P) \in \Pi$, it is enough to show that for a given $\mathcal{M} \subseteq \diamond$ it is decidable whether it is complete or not. For this we have to be able to decide whether a given $(E, P) \in \Pi$ is $\mathcal{M}$-continuable. Let (E, P) be a mosaic on $U \subseteq \alpha$, and let $i \in \alpha$, $c_i\sigma \in \mathrm{Subterm}(\tau)$, $s \in P(c_i\sigma) \cap E$. There are only finitely many such choices of i, σ and s. By Lemma 4.1(i), if (E, P) has an $\mathcal{M}$-like extension (E', P') with $(\exists u)s(i/u) \in P'(\sigma) \cap E'$, then it also has an extension with $U' \subseteq U \cup \{\alpha\}$. Thus we can check all possible mosaics (E', P') on $U \cup \{\alpha\}$ (since there is only a finite number of them) for the properties

(a) $(\exists u)s(i/u) \in P'(\sigma) \cap E'$

(b) $(E, P) \prec (E', P')$

(c) (E', P') is $\mathcal{M}$-like.

All these properties are decidable (since "everything is finite" in the mosaics $(E, P), (E', P')$), whence $\mathcal{M}$-continuability of (E, P) is decidable. ■(Claim 4.2)

Proof of Claim 4.1: The idea here is that models can be replaced by complete sets of mosaics: for all model-valuation pair $\underline{A} \in K$, $k : X \longrightarrow A$ the set of mosaics $\mathcal{M}(\underline{A}, k)$ (which consists of "cut outs" of $\underline{A}, k)$ is complete; and what is even more important is that from all complete set of mosaics $\mathcal{M}$ one can put together an algebra-valuation pair $\underline{A}, k$ such that $\mathcal{M} = \mathcal{M}(\underline{A}, k)$.
(I)Proof of "$K \not\models \tau = 1 \Rightarrow (\exists \mathcal{M} \subseteq \diamond)[\mathcal{M}$ is complete and $\mathcal{M} \not\models \tau]$": Let $\underline{A} \in K$ and let $k : X \longrightarrow A$ be such that $\underline{A} \not\models \tau = 1[k]$. Let $U \stackrel{\text{def}}{=} \mathrm{base}(\underline{A})$, and for all $W \subseteq U$ let

$$\underline{A}(W) \stackrel{\text{def}}{=} (E(W), P(W)),$$

where

$$E(W) \stackrel{\text{def}}{=} 1^{\underline{A}} \cap {}^{\alpha}W,$$

and if $\sigma \in \text{Subterm}(\tau)$, then

$$P(W)\sigma \stackrel{\text{def}}{=} \begin{cases} \sigma^{\underline{A}}[k] \cap {}^{\alpha}W & \text{if } (\forall i)(\forall \eta)\sigma \neq c_i\eta \\ c_i^{[\delta E(W)]}(\sigma^{\underline{A}}[k] \cap {}^{\alpha}W) & \text{if } (\exists \eta)\sigma = c_i\eta. \end{cases}$$

Note that $P(W)\sigma \cap E(W) = \sigma^{\underline{A}}[k] \cap {}^{\alpha}W$ for all $\sigma \in \text{Subterm}(\tau)$. Let $\mathcal{M} \stackrel{\text{def}}{=} \mathcal{M}(\underline{A}, k) \stackrel{\text{def}}{=} \{\underline{A}(\text{Rng}s) : s \in E\}$. We show that $\mathcal{M}$ is a complete set of mosaics and $\mathcal{M} \not\models \tau$. First let us show that $\mathcal{M}$ is a set of mosaics. Let $W \subseteq U$. We show that $\underline{A}(W)$ is a τ, K-mosaic. It is easy to check that $E(W)$ is a K-unit (since K satisfies conditions (b),(d)). Clearly, $P(W) : \text{Subterm}(\tau) \longrightarrow \text{Sb}\delta E(W)$ and $P(W)$ satisfies conditions (2a)...(2d) of Definition 4.2(ii) by the definition of $P(W)$. Thus $\underline{A}(W)$ is a τ, K-mosaic on W. Next we show that $\mathcal{M}$ is complete. Let $W \subseteq U$. Then obviously $\underline{A}(W) \prec \underline{A}(U)$ and $\underline{A}(U)$ is $\mathcal{M}$-like. Moreover, if $i \in \alpha$, $c_i\sigma \in \text{Subterm}(\tau)$ and $s \in P(W)(c_i\sigma) \cap E(W)$ then $(\exists u \in U)s(i/u) \in P(U)(\sigma) \cap E(U)$. Thus $\underline{A}(W)$ is $\mathcal{M}$-continuable, whence it is complete. Finally we show that $\mathcal{M} \not\models \tau$. Since $\underline{A} \not\models \tau = 1$, we have $\tau^{\underline{A}}[k] \neq 1^{\underline{A}}$. Let $s \in 1^{\underline{A}} \setminus \tau^{\underline{A}}[k]$ and $W \stackrel{\text{def}}{=} \text{Rng}s$. Then $s \in E(W) \setminus P(W)(\tau)$, whence $P(W)(\tau) \not\supseteq E(W)$ and so $\mathcal{M} \not\models \tau$. Let $\mathcal{M}' \stackrel{\text{def}}{=} \Pi \cap \mathbf{I}\mathcal{M}$. Then it is easy to check that $\mathcal{M}'$ is complete and $\mathcal{M}' \not\models \tau$. ■(I)

(II)Proof of "$\mathcal{M} \subseteq \Pi$ is complete and $\mathcal{M} \not\models \tau \Rightarrow K \not\models \tau = 1$": Let $\mathcal{M} \subseteq \Pi$ be complete and suppose that $\mathcal{M} \not\models \tau$. Our plan is as follows: Let $M_0 \in \mathcal{M}$ be such that $M_0 \not\models \tau$. Now we construct a mosaic $M_0 \prec M = (E, P)$ step by step starting from M_0 and using Lemma 4.1(iii), which satisfies

$$P(c_i\sigma) \cap E = C_i^{[E]}(P(\sigma) \cap E) \quad \text{for all } c_i\sigma \in \text{Subterm}(\tau).$$

(Here one must make sure of "stepping on" every "defect" at least once.) If $\underline{A} = \underline{Sb}E$ and $k(x) = P(x) \cap E$ for all $x \in X$, then it is easy to see that $\sigma^{\underline{A}}[k] = P(\sigma) \cap E$ for all $\sigma \in \text{Subterm}(\tau)$. Thus $\underline{A} \not\models \tau = 1[k]$ since $M_0 \prec M$. Moreover, $\underline{A} \in K$ since M is a τ, K-mosaic.

Although the full proof can be completely (and almost mechanically) reconstructed from this plan, we sketch it for completeness' sake. If $M = (E, P)$ is a mosaic then $D(M)$ denotes the set of "defects" of M, i.e.

$$D(M) \stackrel{\text{def}}{=} \{(s, i, \sigma) : s \in (P(c_i\sigma) \cap E) \setminus C_i^{[E]}(P(\sigma) \cap E), \quad i \in \alpha, c_i\sigma \in \text{Subterm}(\tau)\}.$$

It is easy to check that

$$(1) \qquad [M = (E, P) \prec M', s \in E \quad \text{and} \quad (s, i, \sigma) \notin D(M)] \Rightarrow (s, i, \sigma) \notin D(M').$$

Next we show that

(2) Every $\mathcal{M}$-like mosaic M can be extended to an $\mathcal{M}$-like mosaic M' such that $D(M) \cap D(M') = 0$.

Indeed, let $\beta \stackrel{\text{def}}{=} |D(M)|$ and let $f : \beta \twoheadrightarrow D(M)$ be an enumeration of $D(M)$. Define the following sequence of mosaics by recursion: $M_0 \stackrel{\text{def}}{=} M$. Let $\gamma < \beta$ and assume that for all $\eta < \gamma$, M_γ is defined such that

$$(*) \qquad \text{for all } \delta < \eta,\ M_\delta \prec M_\eta \text{ and } M_\eta \text{ is } \mathcal{M}\text{-like.}$$

Let $M'_\gamma = \bigcup\{M_\eta : \eta < \gamma\}$. Then by Lemma 4.1(ii) M'_γ is an $\mathcal{M}$-like mosaic and so it is $\mathcal{M}$-continuable by Lemma 4.1(iii). Thus there is an $\mathcal{M}$-like extension M_γ of M'_γ such that $f_\gamma \notin D(M_\gamma)$. It is not hard to show that M_γ satisfies $(*)$. Let $M' = \bigcup\{M_\gamma : \gamma < \beta\}$. Then it is easy to show that M' has the required property. ■(2)

Now we define an ω-sequence of $\mathcal{M}$-like mosaics: Let $M_0 \in \mathcal{M}$ be such that $M_0 \not\models \tau$. Assume that M_n is already defined in such a way that it is $\mathcal{M}$-like and $(\forall m < n) M_m \prec M_n$. Then let M_{n+1} be an $\mathcal{M}$-like extension of M_n with $D(M_n) \cap D(M_{n+1}) = 0$. (There is such an extension by (2)). Let $M \stackrel{\text{def}}{=} \bigcup\{M_n : n \in \omega\} = (E, P)$. Then obviously $M_0 \prec M$ and below we will show

$$(**) \qquad C_i^{[E]}(P(\sigma) \cap E) = P(\mathrm{c}_i\sigma) \cap E \quad \text{for all } \mathrm{c}_i\sigma \in \mathrm{Subterm}(\tau).$$

First we show $P(\mathrm{c}_i\sigma) \cap E \subseteq C_i^{[E]}(P(\sigma) \cap E)$. This is equivalent with $D(M) = 0$. Let $s \in E$. Then $(\exists n \in \omega) s \in E_n$. Then, by the construction and (1), $(s, i, \sigma) \notin D(M_{n+1})$, whence $(s, i, \sigma) \notin D(M)$ by (1) and $M_{n+1} \prec M$. This completes the proof of $D(M) = 0$. It remains to show that $C_i^{[E]}(P(\sigma) \cap E) \subseteq P(\mathrm{c}_i\sigma)$. But this follows from M being a mosaic since condition (2d) gives $P(\sigma) \cap E \subseteq P(\mathrm{c}_i\sigma) = C_i^{[\delta E]} P(\mathrm{c}_i\sigma)$. ■(II) ■(Claim 4.1)

We note that in the above construction we could have striven for using up all elements of $\mathcal{M}$, and in this way we would have ended up with a model-valuation pair $\underline{A}, k$ for which $\mathbf{I}\mathcal{M}(\underline{A}, k) = \mathbf{I}\mathcal{M}$ (disregarding some minor differences).

Theorem 4.2(iii) is an immediate conseqence of Claims 4.1,4.2. Since it is easy to see that the classes Crs_α, D_α and G_α satisfy conditions (a)...(d) of Theorem 4.2(iii), so far we have shown the decidability of EqCrs_α, EqD_α and EqG_α for $\alpha < \omega$. It remains to show that EqCrs_α and EqG_α are decidable even if $\alpha \geq \omega$.

Just as in the previous proof, we treat the case of $\alpha \geq \omega$ by reducing it to the case of $\alpha < \omega$ (cf. Lemma 3.4 in the previous proof). At the same time we will show that this kind of reduction does not work for D_α. Let X be an infinite set (of variable-symbols). Below we write $\mathrm{Tm}(\mathrm{cyl}_\alpha)$ in place of $\mathrm{Tm}_X(\mathrm{cyl}_\alpha)$.

LEMMA 4.2 [5] *Let* $\gamma \subset \alpha$, $2 \leq |\gamma| < \omega$ *and* $\tau \in \mathrm{Tm}(\mathrm{cyl}_\gamma)$.

(i) $\mathbf{Rd}_\gamma \mathbf{I}Crs_\alpha = \mathbf{I}Crs_\gamma$ *and* $Crs_\alpha \models \tau = 1 \iff Crs_\gamma \models \tau = 1$.

(ii) $\mathbf{HSPRd}_\gamma G_\alpha \subset \mathbf{I}G_\gamma$ *but* $G_\alpha \models \tau = 1 \iff G_\gamma \models \tau = 1$ *if* $\mathrm{ind}(\tau) \subset \gamma$.

(iii) $\mathbf{HSPRd}_\gamma D_\alpha \subset \mathbf{I}D_\gamma$ *and for all* $n \leq |\gamma| - 2$ *there is a* $\sigma \in \mathrm{Tm}(\mathrm{cyl}_\gamma)$ *such that* $D_\alpha \models \sigma = 1 \not\Longrightarrow D_\gamma \models \sigma = 1$ *and* $|\gamma \setminus \mathrm{ind}(\sigma)| \geq n$. *Furthermore,* $D_\alpha \models \delta = 1 \not\Longrightarrow G_\gamma \models \delta = 1$ *for some* $\delta \in \mathrm{Tm}(\mathrm{cyl}_\gamma)$.

We note that the last statement of (iii) shows that the condition "$\mathrm{ind}(\tau) \subset \gamma$" cannot be omitted in (ii).

Proof: (I) Proof of $\mathbf{Rd}_\gamma Crs_\alpha \subseteq \mathbf{I}Crs_\gamma$, $\mathbf{Rd}_\gamma D_\alpha \subseteq \mathbf{I}D_\gamma$, $\mathbf{Rd}_\gamma G_\alpha \subseteq \mathbf{I}G_\gamma$ and $Crs_\alpha \subseteq \mathbf{IRd}_\gamma Crs_a$: Here we will only use the assumption $\gamma \subseteq \alpha$.

Let $\underline{A} \in Crs_\alpha$, $V \stackrel{\text{def}}{=} 1^{\underline{A}}$ and $U \stackrel{\text{def}}{=} \mathrm{base}(\underline{A})$. We define the "reduct-function" rd_γ as follows: If $f \in V$, then

$$\mathrm{rb}_\gamma(f) \stackrel{\text{def}}{=} \langle (f_i, f') : i \in \gamma \rangle, \text{ where } f' \stackrel{\text{def}}{=} f \restriction (\alpha \setminus \gamma), \text{ and}$$

$$\mathrm{rd}_\gamma(X) \stackrel{\text{def}}{=} \mathrm{rb}_\gamma^\star X = \{\mathrm{rb}_\gamma(f) : f \in X\} \text{ for } X \subseteq V.$$

It is not hard to show by induction that

$$(*) \qquad \mathrm{rd}_\gamma \quad \text{is an isomorphism between} \quad \underline{Rd_\gamma Sb}V \quad \text{and} \quad \underline{Sb}\mathrm{rd}_\gamma V.$$

(The proof can be found in [9] 4.7.1.2 (p. 191) and in [8] 3.1.125. We note that $\mathrm{rb}_\gamma(f)$ is $f \restriction \gamma$ "colored" by f' to make rd_γ 1-1 and homomorphism w.r.t. $\sim$.) Since $\underline{Rd_\gamma A} \subseteq \underline{Rd_g Sb}V$ (by $\underline{A} \subseteq \underline{Sb}V$) and $\mathrm{rd}_\gamma V$ is a Crs_γ-unit, it follows from $(*)$ that $\mathbf{Rd}_\gamma Crs_\alpha \subseteq \mathbf{I}Crs_\gamma$. To prove the next two statements one has to show that $\mathrm{rd}_\gamma V$ is a D_γ (resp. G_γ) unit if V is a D_γ (resp. G_γ) unit. These are easy to check so we omit the details.

Now let $\underline{A} \in Crs_\gamma$ be arbitrary. We show that $\underline{A} \in \mathbf{IRd}_\gamma Crs_\alpha$. Let $V \stackrel{\text{def}}{=} 1^{\underline{A}}$, $u \notin \mathrm{base}(\underline{A})$, $q \stackrel{\text{def}}{=} \langle u : i \in \alpha \setminus \gamma \rangle$ and for all $x \in A$, $f(x) \stackrel{\text{def}}{=} \{s \cup q : s \in x\}$. It is easy to check that $f : \underline{A} \rightarrowtail \underline{Rd_\gamma Sbf}V$ is a homomorphism and $f^\star A$ is closed under the operations of $\underline{Sbf}V$. Thus $\underline{A} \in \mathbf{IRd}_\gamma Crs_\alpha$. This also completes the proof of $\mathbf{I}Crs_\gamma = \mathbf{Rd}_\gamma \mathbf{I}Crs_\alpha$.

(II)Proof of "$G_\alpha \models \tau = 1 \iff G_\gamma \models \tau = 1$ if $\mathrm{ind}(\tau) \subset \gamma$": The implication $G_\gamma \models \tau = 1 \Longrightarrow G_\alpha \models \tau = 1$ holds for all $\tau \in \mathrm{Tm}(\mathrm{cyl}_\gamma)$ since

$$G_\alpha \not\models \tau = 1 \Longrightarrow \mathbf{Rd}_\gamma G_\alpha \not\models \tau = 1 \Longrightarrow G_\gamma \not\models \tau = 1$$

by $\mathbf{Rd}_\gamma G_\alpha \subseteq \mathbf{I}G_\gamma$. It remains to prove $G_\gamma \not\models \tau = 1 \Longrightarrow G_\alpha \not\models \tau = 1$ if $\mathrm{ind}(\tau) \subset \gamma$. It is enough to show $G_\gamma \not\models \tau = 0 \Longrightarrow G_\alpha \not\models \tau = 0$ for all τ such that

[5] Item (i) was published in [15], Proposition 8(ii), and the proof is cited in the monograph [8], see [8] 5.5.15.

$\mathrm{ind}(\tau) \subset \gamma$. Suppose that $\mathrm{ind}(\tau) \stackrel{\mathrm{def}}{=} H \subset \gamma$ and $G_\gamma \not\models \tau = 0$, say $\underline{A} \not\models \tau = 0[k]$ for some $\underline{A} \in G_\gamma$ and $k : X \longrightarrow A$. Let $p \in \tau^{\underline{A}}[k]$, $U \stackrel{\mathrm{def}}{=} \mathrm{base}(\underline{A})$, $q \stackrel{\mathrm{def}}{=} p \restriction (\gamma \setminus H)$ and let $r \in {}^{(\alpha\setminus H)}\mathrm{Rng}q$ be arbitrary. (The existence of such r follows from $\gamma \setminus H \neq \emptyset$.) Let $V \stackrel{\mathrm{def}}{=} \{s \in {}^H U : s \cup q \in 1^{\underline{A}}\}$ and $W \stackrel{\mathrm{def}}{=} \bigcup\{{}^\alpha\mathrm{Rng}(s \cup q) : s \in V\}$. Then W is clearly a G_α-unit. We show that

$$(*) \qquad V = \{s \in {}^H U : s \cup r \in W\}.$$

Indeed, let $s \in {}^H U$. First suppose that $s \cup r \in W$. Then, by the definition of W, $\mathrm{Rng}(s \cup r) \subseteq \mathrm{Rng}(z \cup q)$ for some $z \in V$. Then $z \cup q \in 1^{\underline{A}}$ by the definition of V, and $\mathrm{Rng}(s \cup q) \subseteq \mathrm{Rng}(z \cup q)$. Then $s \cup q \in 1^{\underline{A}}$ since $1^{\underline{A}}$ is a G_γ-unit, and thus $s \in V$. Now suppose that $s \in V$. Then $s \cup r \in W$ since $\mathrm{Rng}(s \cup r) \subseteq \mathrm{Rng}(s \cup q)$. This completes the proof of $(*)$.

Let $\underline{C} \stackrel{\mathrm{def}}{=} \underline{SbV}$ and $\underline{B} \stackrel{\mathrm{def}}{=} \underline{SbW}$. Define

$$f \stackrel{\mathrm{def}}{=} \langle \{s \in {}^H U : s \cup q \in a\} : a \in A \rangle, \text{ and}$$

$$g \stackrel{\mathrm{def}}{=} \langle \{s \in {}^H U : s \cup r \in b\} : b \in B \rangle.$$

Then $f(1^{\underline{A}}) = V$ by definition, and $g(W) = V$, moreover, $\mathrm{Rng}g = \mathrm{Sb}V$ by $(*)$. It is not hard to check that $f : \underline{Rd_H A} \longrightarrow \underline{C}$ and $g : \underline{Rd_H B} \twoheadrightarrow \underline{C}$ are homomorphisms (the proof can be found e.g. in [8] 3.1.117) and $f(\tau^{\underline{A}}[k]) \neq 0$. Let $k'(x) \stackrel{\mathrm{def}}{=} fk(x)$ and let $k''(x) \in B$ be such that $gk''(x) = k'(x)$ for all $x \in X$. Then $\tau^{\underline{C}}[k'] = f(\tau^{\underline{A}}[k]) = g(\tau^{\underline{B}}[k''])$, so $\tau^{\underline{B}}[k''] \neq 0$. Since $\underline{B} \in G_\alpha$, this proves $G_\alpha \not\models \tau = 0$. ■(II)

$(*)$ is a crucial part of the above proof. We give an example showing that $(*)$ (with the necessary modifications) fails for the classes D_α: Let $H \subset n \in \alpha = \omega$, $p \stackrel{\mathrm{def}}{=} \mathrm{Id} \restriction n$, $1^{\underline{A}} = \{p\} \cup \bigcup\{\mathrm{D}_{\mathrm{ij}}^{[{}^n\omega]} : \mathrm{i} < \mathrm{j} < \mathrm{n}\}$ and let $r \in {}^{(\alpha\setminus H)}\omega$ be arbitrary. Then $1^{\underline{A}}$ is a D_n-unit and p is the only repetition-free sequence in $1^{\underline{A}}$. Let $q \stackrel{\mathrm{def}}{=} p \restriction (n \setminus H)$, $V \stackrel{\mathrm{def}}{=} \{s \in {}^H\omega : s \cup q \in 1^{\underline{A}}\}$ and let W be the smallest D_α-unit containing $\{s \cup r : s \in V\}$. Then $V \neq \{s \in {}^H\omega : s \cup r \in W\}$. This is because there is a repetition-free sequence $z \in {}^H\omega$ with $z \neq \mathrm{Id} \restriction H$ and $z \cup r \in W$ (and then $z \notin V$): if r is not repetition-free then since all sequences $z \cup r$, $z \in {}^H H$ can be obtained from $r \cup (\mathrm{Id} \restriction H)$ using the "auxiliary storage places" provided by the repetition (see the proof of $(***)$ in (IV) below); if r is repetition-free then because $\mathrm{Rng}r$ is infinite.

(III)Proof of $G_\gamma \not\subseteq \mathbf{HSPRd}_\gamma G_\alpha$, $D_\gamma \not\subseteq \mathbf{HSPRd}_\gamma D_\alpha$ and $(\exists\delta \in \mathrm{Tm}(\mathrm{cyl}_\gamma))[D_\alpha \models \delta = 1, G_\gamma \not\models \delta = 1]$: It suffices to prove the last statement since it implies the previous ones: Let $\underline{A} \in G_\gamma \subseteq D_\gamma$ be such that $\underline{A} \not\models \delta = 1$. Then $\underline{A} \notin \mathbf{HSPRd}_\gamma D_\alpha \supseteq \mathbf{HSPRd}_\gamma G_\alpha$ since $D_\alpha \models \delta = 1$.

Suppose (for simplicity) that $\gamma = n \in \omega$. Let $\bar{d} \stackrel{\mathrm{def}}{=} \prod\{-\mathrm{d}_{ij} : i < j < n\}$ and let

$$\delta_{ij}(x) \stackrel{\mathrm{def}}{=} x \cdot \bar{d} - \mathrm{c}_i(\bar{d} \cdot \mathrm{c}_j(\mathrm{d}_{ij} \cdot \mathrm{c}_i(-\mathrm{d}_{ij} \cdot \mathrm{c}_j(x))))$$

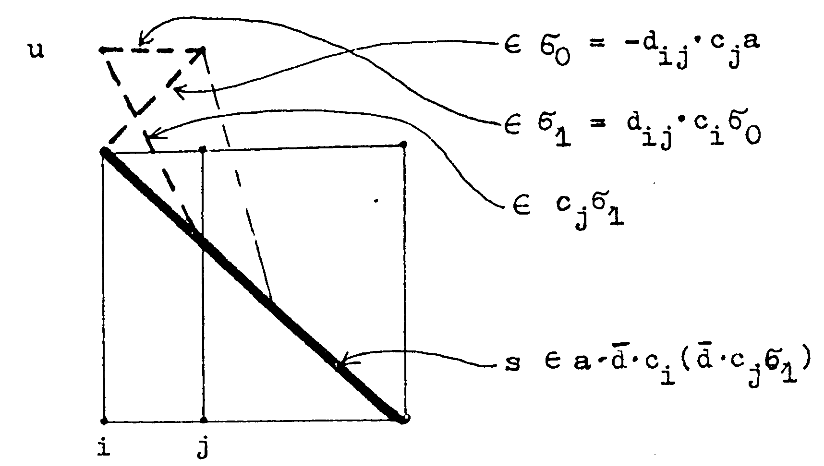


Figure 3.

for all $i, j \in n$. Let $y_0, y_1, \ldots, y_n$ be $n+1$ different variable-symbols and let

$$\tau_i \stackrel{\text{def}}{=} y_i \cdot \prod\{-y_j : j \in (n+1) \setminus \{i\}\}$$

for all $i \leq n$. Let

$$\delta \stackrel{\text{def}}{=} \delta_{01}(\tau_n) \cdot \prod\{\mathsf{c}_i \delta_{if(i)}(\tau_i) : i \in n\},$$

where $f(i) \stackrel{\text{def}}{=} i+1 \pmod{n}$ if $i \in n$. We show that $G_n \not\models \delta = 0$ while $D_\alpha \models \delta = 0$ if $n \subset \alpha$. First we show $D_\alpha \models \delta = 0$. Let $i, j \in \alpha$, $i \neq j$, $\underline{A} \in Crs_\alpha$, $a \in A$ and $s \in 1^{\underline{A}}$. Then the following is easy to check (see Figure 3):

(∗) If $s \in \delta_{ij}^{\underline{A}}(a)$, then there is no $u \in \text{base}(\underline{A}) \setminus \text{Rng}(s \restriction n)$ with $\{s(i/u), s(j/u), s(i/u)(j/u)\} \subseteq 1^{\underline{A}}$.

Let $\underline{A} \in D_\alpha$ and $k : X \longrightarrow A$. Let $a_i \stackrel{\text{def}}{=} \tau_i^{\underline{A}}[k]$. Then $\{a_i : i \leq n\}$ consists of pairwise disjoint elements. Suppose that $s \in \delta_{01}(a_n) \cdot \prod\{\mathsf{c}_i \delta_{ifi}(a_i) : i \in n\}$. Let $k \in \alpha \setminus n$, $u \stackrel{\text{def}}{=} s(k)$. Suppose that $u \notin \text{Rng}(s \restriction n)$. Then $s(0/u), s(1/u), s(0/u)(1/u) \in 1^{\underline{A}}$ since $\underline{A} \in D_\alpha$ and this contradicts $s \in \delta_{01}(a_n)$ by (∗). Then there is an $i \in n$ with $u = s(i)$. If $j \in n \setminus \{i\}$, then $u \neq s_j$ since $s \restriction n$ is repetition-free by $s \in \delta_{01}(a_n)$. Then there is a w such that $z \stackrel{\text{def}}{=} s(i/w) \in \delta_{ifi}(a_i)$ since $s \in \mathsf{c}_i \delta_{ifi}(a_i)$. Now $u \neq w$ since $s = s(i/u) \in a_n$, $s(i/w) \in a_i$ and $a_n \cap a_i = \emptyset$. Then $u \notin \text{Rng}(z \restriction n)$, contradicting $z \in \delta_{ifi}(a_i)$ by (∗) as before. With this we have shown $D_\alpha \models \delta = 0$.

Next we construct an $\underline{A} \in G_n$ with $\underline{A} \not\models \delta = 0$. For all $i \in n$ let $u_i = n + i$ and let

$$V \stackrel{\text{def}}{=} {}^n n \cup \bigcup \{ {}^n([n \cup \{u_i\}] \setminus \{i\}) : i \in n \},\ \underline{A} \stackrel{\text{def}}{=} \underline{SbV},$$

$$s \stackrel{\text{def}}{=} \mathrm{Id} \restriction n,\ a_n \stackrel{\text{def}}{=} \{s\},$$

$$a_i \stackrel{\text{def}}{=} \{s(i/u_i)\} \text{ if } i \in n \text{ and}$$

$$k(y_i) \stackrel{\text{def}}{=} a_i \text{ if } i \leq n.$$

We prove $\underline{A} \not\models \delta = 0[k]$ by showing that $s \in \delta^{\underline{A}}[k]$. It is easy to check that $\tau_i^{\underline{A}}[k] = a_i$ for all $i \leq n$. So it is enough to show that

$$s \in \delta_{01}(a_n) \quad \text{and} \quad s(i/u_i) \in \delta_{ifi}(a_i) \quad \text{for all} \quad i \in n.$$

Note that

$$(**) \qquad (\forall i \in n)(\forall z \in 1^{\underline{A}})\{i, n_i\} \not\subseteq \mathrm{Rng} z.$$

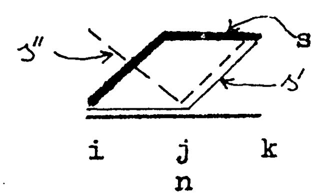
Clearly $s \in a_n \cdot \bar{d}$ and $s(i/u_i) \in a_i \cdot \bar{d}$. Suppose that $s \in \mathsf{c}_0(\bar{d} \cdot \mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0(-\mathsf{d}_{01} \cdot \mathsf{c}_1 a_n)))$. Then there are v and w such that $s^0_v \in \bar{d}$, $v \neq w$ and $s^{01}_{wv} \in \mathsf{c}_1 a_n$. By $(**)$ $s^0_v \in \bar{d}$ gives $v \in \{0, u_0\}$, and $s^{01}_{wv} \in \mathsf{c}_1 a_n = \mathsf{c}_1\{s\}$ gives $w = 0$. Thus $v = u_0$ since $w \neq v$, but then $s^{01}_{wv} \in 1^{\underline{A}}$ contradicts $(**)$. We have shown $s \in \delta_{01}(a_n)$. The proof of $z = s(i/u_i) \in \delta_{ij}(a_i)$ (where $j = fi$) is completely analogous: Suppose that $z \in \mathsf{c}_i(\bar{d} \cdot \mathsf{c}_j(\mathsf{d}_{ij} \cdot \mathsf{c}_i(-\mathsf{d}_{ij} \cdot \mathsf{c}_j a_i)))$. Then $z^i_v \in \bar{d}$, $v \neq w$ and $z^{ij}_{wu} \in \mathsf{c}_j a_i$ for some v, w. $z^i_v \in \bar{d}$ gives $v \in \{i, u_i\}$ and $z^{ij}_{wu} \in \mathsf{c}_j a_i$ gives $w = u_i$. Then $v = i$ and then $z^{ij}_{wu} \in 1^{\underline{A}}$ contradicts $(**)$. ■(III)

Since $\mathbf{I}G_\gamma$ and $\mathbf{I}D_\gamma$ are varieties (see the remark after Definition 4.1), (III) implies the first statements of Lemma 4.2(ii),(iii).

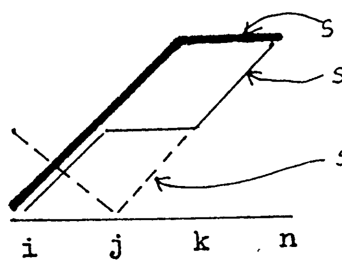
(IV)Proof of "$D_\alpha \models \sigma = 0$ and $D_\gamma \not\models \sigma = 0$ for some $\sigma \in \mathrm{Tm}(\mathrm{cyl}_2)$": Let $\tau(x) \stackrel{\text{def}}{=} x - \mathsf{d}_{01} \cdot \mathsf{c}_1(\mathsf{d}_{01} - \mathsf{c}_0(-\mathsf{d}_{01} \cdot \mathsf{c}_1(\mathsf{d}_{01} \cdot \mathsf{c}_0 x)))$. We show that

$(***)$ If $\underline{A} \in D_\alpha, a \in A$ and $s \in \tau^{\underline{A}}(a)$ then $s(0/s_1)(1/s_0) \notin 1^{\underline{A}}$ and s is repetition-free.

First we show that $s(i/s_j)(j/s_i) \in 1^{\underline{A}}$ for all $i, j \in \alpha$, provided s is not repetition-free. Suppose that $s(n) = s(k)$, $n \neq k$. If $\{i, j\} = \{n, k\}$ then we are done. Suppose first that $n \in \{i, j\}$, say $n = j$, and $k \notin \{i, j\}$.

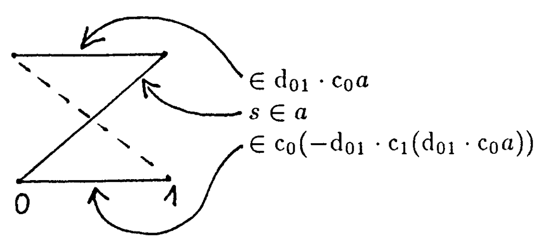


Let $s' \stackrel{\text{def}}{=} s(j/s_i)$ and $s'' \stackrel{\text{def}}{=} s'(i/s'_k)$. Then $s', s'' \in 1^{\underline{A}}$ since $\underline{A} \in D_\alpha$ and clearly $s'' = s(i/s_j)(j/s_i)$. Now suppose that $\{i, j\} \cap \{k, n\} = 0$.



Let $s' \stackrel{\text{def}}{=} s(k/s_j)$, $s'' \stackrel{\text{def}}{=} s'(i/s'_j)(j/s'_i)$ and $z \stackrel{\text{def}}{=} s''(k/s''_n)$. Then $s', s'', z \in 1^{\underline{A}}$ (using the proof of the previous case) and $z = s(i/s_j)(j/s_i)$. We have shown that if a D_α-unit contains a sequence with repetitions then it also contains all its transpositions.

So it is enough to show that $s \in \tau^{\underline{A}}(a)$ implies $s(0/s_1)(1/s_0) \notin 1^{\underline{A}}$ (since, by the above, this implies that s must be repetition-free). Let $s \in -\mathrm{d}_{01} \cdot a$ and suppose that $z = s(0/s_1)(1/s_0) \in 1^{\underline{A}}$. Then the picture below shows that $s \notin \tau^{\underline{A}}(a)$.



With this we have proved $(***)$.

In the remaining part of the proof we assume $\gamma \stackrel{\text{def}}{=} n \in \omega$ for simplicity. Let $y_0 \dots y_{n-2}$ be $n-1$ distinct variable symbols. Let

$$\tau_i \stackrel{\text{def}}{=} y_i \text{ if } i \in n-1,$$

$$\tau_{n-1} \stackrel{\text{def}}{=} \prod\{-y_i : i \in n-1\} \cdot \mathrm{c}_1 y_1,$$

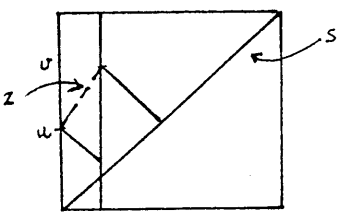
$$\delta \stackrel{\text{def}}{=} \tau(y_1) - \mathrm{c}_0\mathrm{c}_1(-\mathrm{d}_{01} \cdot \sum\{\mathrm{c}_0\tau_i \cdot \mathrm{c}_1(\mathrm{d}_{01} \cdot \mathrm{c}_0\tau_i) : i \in n\}) \cdot y_1$$

We show that $D_\alpha \models \delta = 0$ if $n \subset \alpha$ and $D_n \not\models \delta = 0$. Let $\underline{A} \in D_\alpha$, $k : X \longrightarrow A$ and suppose that $s \in \delta^{\underline{A}}[k]$. Then s is repetition-free by $(***)$ whence $|\mathrm{Rng} s| > n$. For all $i \in n$ let

$$a_i \stackrel{\text{def}}{=} \tau_i^{\underline{A}}[k] \text{ and}$$

$$H_i \stackrel{\text{def}}{=} \{u \in \mathrm{Rng} s : s(1/u) \in a_i\}.$$

Then $\bigcup\{H_i : i \in n\} = \mathrm{Rng}s$ and, since $|\mathrm{Rng}s| > n$, there is an $i \in n$ such that $|H_i| \geq 2$. Let $i \in n$ and $u, v \in H_i$, $u \neq v$. Let $k, l \in n$ be such that $k < l$ and $s(k) = u$, $s(l) = v$.



Then $s(0/u) \in 1^{\underline{A}}$, and $z \stackrel{\text{def}}{=} s(0/u)(1/v) \in 1^{\underline{A}}$ since $l > 0$; moreover, $s \in \mathrm{c}_0\mathrm{c}_1(-\mathrm{d}_{01} \cdot \{z\})$. We show that $z \in \mathrm{c}_0 a_i \cdot \mathrm{c}_1(\mathrm{d}_{01} \cdot \mathrm{c}_0 a_i)$, contradicting $s \in \delta^{\underline{A}}[k]$. $z(0/s_0) = s(1/v) \in a_i$ since $v \in H_i$ and so $z \in \mathrm{c}_0 a_i$. And $z(1/u) \in \mathrm{d}_{01} \cdot \mathrm{c}_0 a_i$ since $z(1/u)(0/s_0) = s(1/u) \in a_i$ by $u \in H_i$. With this we have shown $D_\alpha \models \delta = 0$ if $\alpha > n$.

Now we construct an $\underline{A} \in D_n$ and $k : X \longrightarrow A$ such that $\underline{A} \not\models \delta = 0[k]$. Let $s \stackrel{\text{def}}{=} \mathrm{Id} \restriction n$, $V \stackrel{\text{def}}{=} \{s\} \cup \bigcup\{D_{ij}^{[^n n]} : i < j < n\}$, $\underline{A} \stackrel{\text{def}}{=} \underline{SbV}$ and $k(y_i) \stackrel{\text{def}}{=} \{s(1/i)\}$ for all $i \in n - 1$. We show that $s \in \delta^{\underline{A}}[k]$. Let $a_i \stackrel{\text{def}}{=} \tau_i^{\underline{A}}[k]$ and $H_i \stackrel{\text{def}}{=} \{u \in \mathrm{Rng}s : s(1/u) \in a_i\}$ if $i \in n$. Then $a_i = \{s(1/i)\}$ and $H_i = \{i\}$ for all $i \in n$. It is easy to check that $s \in \tau(a_1)$ since $a_1 = \{s\}$. Also, since $|H_i| < 2$, we have $s \notin \mathrm{c}_0\mathrm{c}_1(-\mathrm{d}_{01} \cdot \mathrm{c}_0 a_i \cdot \mathrm{c}_1(\mathrm{d}_{01} \cdot \mathrm{c}_0 a_i))$ for all $i \in n$. Thus $s \in \delta^{\underline{A}}[k]$. ■(Lemma4.2)

Clearly, Theorem 4.2(iii) and Lemma 4.2(i) (Lemma 4.2(ii)) give a decision-algorithm for $\mathrm{Eq}Crs_\alpha$ ($\mathrm{Eq}G_\alpha$) if $\alpha = \omega$. ■(Theorem 4.2)

THEOREM 4.3 (i) $\mathrm{Eq}G_\alpha \neq \mathrm{Eq}D_\alpha$ *if* $\alpha \geq 2$.

(ii) $\mathrm{Eq}D_\alpha \neq \mathrm{Eq}\mathbf{F}D_\alpha$ *if* $\alpha \geq \omega$.

Proof: Let $\tau(x) \in \mathrm{Tm}(\mathrm{cyl}_2)$ be the term defined in (IV) of the proof of Lemma 4.2. There we showed that if $\underline{A} \in D_\alpha$, $a \in A$ and $s \in \tau^{\underline{A}}(a)$ then

(a) $s(0/s_1)(1/s_0) \notin 1^{\underline{A}}$ and

(b) s is repetition-free.

Now $G_\alpha \models \tau(x) = 0$ follows from (a), and $\mathbf{F}D_\alpha \models \tau(x) = 0$ if $\alpha \geq \omega$ follows from (b) (since if $s \in 1^{\underline{A}}$ is repetition-free and $\underline{A} \in D_\alpha$ then $d_{0i}^{\underline{A}} \neq d_{0j}^{\underline{A}}$ for all $0 < i < j < \alpha$, so $|A| \geq \omega$). But it is easy to construct an $\underline{A} \in D_\alpha$ (for all $\alpha \geq 2$) with $\underline{A} \not\models \tau(x) = 0$, namely the one given in (IV) is such. ■(Theorem 4.3)

REMARK 4.3 (i) We suspect the following difference between D_α and G_α: Let $\alpha < \omega$. If a new coordinate can be added to $\underline{A} \in G_\alpha$ then arbitrary many new coordinates can be added to it, that is, $\mathbf{ISRd}_\alpha G_{\alpha+1} = \mathbf{ISRd}_\alpha G_{\alpha+\beta}$ for arbitrary $\beta \geq 1$. (This is probably provable by the methods used in part (II)

of the proof of Lemma 4.2.) By contrast, for all $n \in \omega$ there is an $\underline{A} \in D_\alpha$ such that n coordinates can, but $n+1$ coordinates cannot be added to it, that is, $\mathbf{ISRd}_\alpha D_{\alpha+n} \neq \mathbf{ISRd}_\alpha D_{\alpha+n+1}$. (For example it can be shown that the α-reduct of $\underline{A} \in D_\gamma$ ($\gamma = \alpha+n$) constructed in part (IV) of the proof of Lemma 4.2 is not in $\mathbf{ISRd}_\alpha D_{\alpha+n+1}$.)
(ii) If $\alpha \leq 2$ then Crs_α is strongly decidable since $\mathbf{I}Crs_\alpha = WCA_\alpha$ in this case. It is quite plausible that using the methods of Remark 3.4 one can show that G_2 and D_2 are also strongly decidable. From now on let $\alpha \geq 3$. We don't know whether Crs_α is strongly decidable or not. We don't even know whether $\mathrm{Eq}Crs_\alpha = \mathrm{Eq}\mathbf{F}Crs_\alpha$ or whether the word problem for Crs_α is solvable. (The same problems are open for D_α and G_α, except for Theorem 4.3(ii).) We don't know whether $\mathrm{Eq}D_\alpha$ is decidable for $\alpha \geq \omega$. Richard Thompson gave a nice finite scheme axiomatization for $\mathrm{Eq}D_\alpha$ based on the present work. ■

One could call K-units satisfying conditions (a), (b) of Theorem4.2 "loosely connected". Below we define the "opposite" of this property, called the "patchwork property", and show that the equational theory of Crs_α's with units having the patchwork property is no longer decidable.

DEFINITION 4.6 A Crs_α-unit V is said to have the patchwork property (V is a P_α-unit) if

$$(\forall s, z \in V)(\forall H \subseteq \alpha)[(s \restriction H) \cup z \restriction (\alpha \setminus H)] \in V.$$

Let $P_\alpha \stackrel{\text{def}}{=} \{\underline{A} \in Crs_\alpha : 1^{\underline{A}}$ has the patchwork property $\}$. ■

Experience suggests that it is the patchwork property the lack of which is responsible for the behaviour of the classes Crs_α, D_α and G_α. Furthermore, there is a close connection between this property and the axiom C_4. This is what the next lemma is about.

LEMMA 4.3 (i) $[\underline{Sb}V \models C_4^\alpha \Leftrightarrow \underline{Sb}V \in \mathbf{P}P_\alpha]$ *if V is a Crs_α-unit. But if $\alpha \geq 3$ then there is an $\underline{A} \in Crs_\alpha$ with $\underline{A} \models C_4^\alpha$ and $\underline{A} \notin \mathbf{HSP}P_\alpha$.*

(ii) $\mathbf{HSP}P_\alpha = \mathbf{SP}P_\alpha = \mathbf{I}\{\underline{A} \in Crs_\alpha : 1^{\underline{A}}$ *is a union of P_α-units with disjoint bases* $\}$ *and $\mathrm{Eq}(P_\alpha)$ is not decidable (and is not finitely based) if $\alpha \geq 3$.*

(iii) $\mathbf{SP}(P_\alpha \cap D_\alpha) = \mathbf{SP}Cs_\alpha = RCA_\alpha$.

Proof: First we prove the first part of (i).

Let V be a Crs_α-unit. For $s \in V$ let

$$\mathrm{zd}(s) \stackrel{\text{def}}{=} \bigcup\{\mathrm{C}_{i_0}^{[V]} \ldots \mathrm{C}_{i_n}^{[V]}\{s\} : n \in \omega, i \in {}^{n+1}\alpha\},$$

and let $\mathrm{Subu}(V) \stackrel{\text{def}}{=} \{zd(s) : s \in V\}$[6]. Then it is not hard to show[7] that $\underline{Sb}V$ is isomorphic to a direct product of algebras in $\{\underline{Sb}W : W \in \mathrm{Subu}(V)\}$. Thus

[6]The set of subunits.
[7]Or one can use [8] 3.1.76

it is enough to show that for all $s \in V$, $\mathrm{zd}(s)$ has the patchwork property if $\underline{Sb}V \models C_4^\alpha$. Throughout this proof we write C_i instead of $\mathrm{C}_i^{[V]}$ for all $i \in \alpha$.

Suppose that $\underline{Sb}V \models C_4^\alpha$. We show that

$$(*) \qquad \{s, s(i/u)\} \subseteq V \Longrightarrow [s(j/w) \in V \Longleftrightarrow s(i/u)(j/w) \in V].$$

Indeed, assume $\{s, s(i/u)\} \subseteq V$. If $i = j$ then we are done. So suppose that $i \neq j$. If $z \stackrel{\text{def}}{=} s(i/u)(j/w) \in V$ then $z \in \mathrm{C}_j\mathrm{C}_i\{s\} = \mathrm{C}_i\mathrm{C}_j\{s\}$, hence $s(j/w) \in V$. If $s(j/w) \in V$ then $s(i/u) \in \mathrm{C}_i\mathrm{C}_j\{s(j/w)\} = \mathrm{C}_j\mathrm{C}_i\{s(j/w)\}$ so $s(i/u)(j/w) \in V$. $\blacksquare(*)$

Let $Z \subseteq W \subseteq V$. Z is said to have the patchwork property *in* W if

$$(\forall s, z \in Z)(\forall H \subseteq \alpha)[s \restriction H \cup z \restriction (\alpha \setminus H)] \in W.$$

Z is *good* if

$$(\forall i \in \alpha)(\forall z \in \mathrm{C}_i Z)[\{z\} \cup Z \quad \text{has the patchwork property in } \mathrm{C}_i Z].$$

Let $s \in V$. Clearly, $\{s\}$ is good. We claim that

$$(**) \qquad \text{if } Z \subseteq V \text{ is good, then so is } \mathrm{C}_i Z \text{ for all } i \in \alpha.$$

Let $j \in \alpha$. We have to show that

$$(\forall p \in \mathrm{C}_j\mathrm{C}_i Z)(\{p\} \cup \mathrm{C}_i Z \quad \text{has the patchwork property in} \quad \mathrm{C}_j\mathrm{C}_i Z).$$

It suffices to show that

$$(\forall p \in \mathrm{C}_j\mathrm{C}_i Z)(\forall q \in \mathrm{C}_i Z)(\forall H \subseteq \alpha)[p \restriction H \cup q \restriction (\alpha \setminus H)] \in \mathrm{C}_j\mathrm{C}_i Z.$$

Let $p \in \mathrm{C}_j\mathrm{C}_i Z$, $q \in \mathrm{C}_i Z$, $H \subseteq \alpha$ and $G \stackrel{\text{def}}{=} \alpha \setminus H$. Then $p = z(i/u)(j/w)$ and $q = s(i/v)$ for some $z, s \in Z$ and u, v, w such that $z(i/u) \in V$ and then $z(j/w) \in V$ by $(*)$. Let $g \stackrel{\text{def}}{=} z(i/u)(j/w) \restriction H \cup s(i/v) \restriction G$ and $f \stackrel{\text{def}}{=} z \restriction H \cup s \restriction G$. Then $f \in \mathrm{C}_i Z$ since Z is good, $z \in Z$ and $s \in \mathrm{C}_i Z$. If $\{i, j\} \subseteq G$ then $g = z \restriction H \cup s(i/v) \restriction G \in \mathrm{C}_i Z$ since Z is good. Similarly, if $i \in H$ and $j \in G$ then $g = z(i/u) \restriction H \cup s \restriction G \in \mathrm{C}_i Z$ since Z is good and $z(i/u) \in \mathrm{C}_i Z$. Now suppose that $j \in H$ and $i \in G$. Then $g = z(j/w) \restriction H \cup s(i/v) \restriction G = f(j/w)(i/v)$. Moreover, $f \in V$, $f(j/w) = z(j/w) \restriction H \cup s \restriction G \in V$ and $f(i/v) = z \restriction H \cup s(i/v) \restriction G \in V$ since Z is good and so $(*)$ gives $f(j/w)(i/v) \in V$ whence $g \in \mathrm{C}_j\mathrm{C}_i Z$. Finally, suppose that $\{i, j\} \subseteq H$. Then similarly, $f, f(i/u), f(j/w) \in V$ since $z, z(i/u), z(j/w) \in V$, $s \in Z$ and Z is good, so $(*)$ gives $g = f(i/u)(j/w) \in V$ whence $g \in \mathrm{C}_j\mathrm{C}_i Z$. $\blacksquare(**)$

Now we are ready to show that $\mathrm{zd}(s)$ has the patchwork property. Let $p, q \in \mathrm{zd}(s)$ and $H \subseteq \alpha$. Then $p \in \mathrm{C}_{i_1} \ldots \mathrm{C}_{i_n}\{s\}$ and $q \in \mathrm{C}_{j_1} \ldots \mathrm{C}_{j_k}\{s\}$ for some $i_1 \dots i_n, j_1 \dots j_k \in \alpha$ and so $p, q \in Z \stackrel{\text{def}}{=} \mathrm{C}_{i_1} \dots \mathrm{C}_{i_n} \mathrm{C}_{j_1} \dots \mathrm{C}_{j_k}\{s\}$. Since $\{s\}$ is good, Z is good by $(**)$ (and induction). Then $[p \restriction H \cup q \restriction (\alpha \setminus H)] \in \mathrm{C}_{i_1} Z \subseteq \mathrm{zd}(s)$. With this we have shown that $\underline{Sb}V \in \mathbf{P}P_\alpha$ provided $\underline{Sb}V \models C_4$. The other direction is easily shown by checking that if V has the patchwork property then $\underline{Sb}V \models C_4^\alpha$.

Proof of the remaining parts: V is said to be a Gp_α-unit if it is the union of P_α-units with disjoint bases, that is, if there is a system $\langle U_j : j \in J\rangle$ of pairwise disjoint sets such that $V = \bigcup\{V \cap {}^\alpha U_j : j \in J\}$ and $V \cap {}^\alpha U_j$ has the patchwork property for all $j \in J$. Let

$$Gp_\alpha \stackrel{\text{def}}{=} \{\underline{A} \in Crs_\alpha : 1^{\underline{A}} \text{ is a } Gp_\alpha\text{-unit}\}.$$

Using (the easily provable) [8] 3.1.76 it is easy to show that $\mathbf{SP}P_\alpha = \mathbf{I}Gp_\alpha$. To prove $\mathrm{Eq}P_\alpha = \mathbf{I}Gp_\alpha$ we have to show $\mathbf{H}Gp_\alpha \subseteq \mathbf{I}Gp_\alpha$. The proof goes like that of $\mathbf{H}Crs_\alpha \subseteq \mathbf{I}Crs_\alpha$, one only has to add that $\mathrm{Rep}(F, c)V$ is a Gp_α-unit if V is a Gp_α-unit (this is not hard to check, for example analogously to the proof of [8] 3.1.91).

Now we begin the proof of the undecidability of $\mathrm{Eq}P_\alpha$. Let $\tau \in \mathrm{Tm}(\mathrm{cyl}_3)$. We define (recursively) a $\tau' \in \mathrm{Tm}(\mathrm{cyl}_3)$ such that

$$Cs_3 \models \tau = 0 \iff P_\alpha \models \tau' = 0.$$

Since $\mathrm{Eq}Cs_3$ is undecidable (see [8] 4.2.18), this will prove the undecidability of $\mathrm{Eq}P_\alpha$. Let $\delta \stackrel{\text{def}}{=} \Pi\{\mathrm{c}_i \mathrm{d}_{ij} : i, j \in 3\}$. Define the function $\mathrm{tr} : \mathrm{Tm}(\mathrm{cyl}_3) \longrightarrow \mathrm{Tm}(\mathrm{cyl}_3)$ as follows:

$$\mathrm{tr}(y) \stackrel{\text{def}}{=} \delta \cdot y \text{ if } y \in X$$

$$\mathrm{tr}(\mathrm{d}_{ij}) \stackrel{\text{def}}{=} \delta \cdot \mathrm{d}_{ij} \text{ if } i, j \in 3$$

$$\mathrm{tr}(\sigma \cdot \eta) \stackrel{\text{def}}{=} \mathrm{tr}(\sigma) \cdot \mathrm{tr}(\eta) \text{ if } \sigma, \eta \in \mathrm{Tm}(\mathrm{cyl}_3)$$

$$\mathrm{tr}(-\sigma) \stackrel{\text{def}}{=} \delta - \mathrm{tr}(\sigma) \text{ if } \sigma \in \mathrm{Tm}(\mathrm{cyl}_3)$$

$$\mathrm{tr}(\mathrm{c}_i\sigma) \stackrel{\text{def}}{=} \delta \cdot \mathrm{c}_i \mathrm{tr}(\sigma) \text{ if } i \in 3 \text{ and } \sigma \in \mathrm{Tm}(\mathrm{cyl}_3).$$

We show that $Cs_3 \not\models \tau = 0 \iff P_\alpha \not\models \tau' = 0$. Let $\underline{A} \in P_\alpha$, $k : X \longrightarrow A$ and $s \in \mathrm{tr}(\tau)^{\underline{A}}[k]$. Let $q \stackrel{\text{def}}{=} s \restriction (\alpha \setminus 3)$, $V \stackrel{\text{def}}{=} \{z : z \cup q \in 1^{\underline{A}}\}$ and $W \stackrel{\text{def}}{=} \{z : z \cup q \in \delta^{\underline{A}}[k]\}$. Since $\underline{A} \in P_\alpha$, $V = H \times G \times K$ for some sets H, G, K and one can check that $W = {}^3U$ where $U = H \cap G \cap K$. Then the following is not hard to show (by induction on τ): for all $z \in V$

$$z \cup q \in \mathrm{tr}(\tau)^{\underline{A}}[k] \Leftrightarrow z \in \mathrm{tr}(\tau)^{\underline{Sb}V}[k'] \Leftrightarrow z \in \tau^{\underline{Sb}^3U}[k''],$$

where

$$k'(y) \stackrel{\text{def}}{=} \{z \in V : z \cup q \in k(y)\}$$

and

$$k''(y) \stackrel{\text{def}}{=} \{z \in {}^3U : z \cup q \in k(y)\}$$

for all $y \in X$. Thus $s \in \mathrm{tr}(\tau)^{\underline{A}}[k] \Rightarrow s \restriction 3 \in \tau^{\underline{Sb}^3 U}[k'']$, that is, $Cs_3 \not\models \tau = 0$. Conversely, it is easy to see that $Cs_3 \subseteq \mathbf{ISRd}_3 Cs_3$ (see e.g. [8] 3.1.121) and since $Cs_3 \models \delta = 1$, we have $Cs_3 \models \tau = \mathrm{tr}(\tau)$ and thus $Cs_3 \not\models \tau = 0 \Rightarrow Cs_3 \not\models \mathrm{tr}(\tau) = 0 \Rightarrow Cs_\alpha \not\models \mathrm{tr}(\tau) = 0 \Rightarrow P_\alpha \not\models \mathrm{tr}(\tau) = 0$ since $Cs_\alpha \subseteq P_\alpha$. With this we have shown that $\mathrm{Eq}P_\alpha$ is undecidable.

A slight modificaton of the above proof immediately yields $\mathrm{Eq}(P_\alpha \cap D_\alpha) = \mathrm{Eq}(Cs_\alpha)$ which in turn implies (iii). Since $P_\alpha \cap D_\alpha = \{\underline{A} \in P_\alpha : \underline{A} \models \mathrm{c}_i \mathrm{d}_{ij} = 1 \text{ for all } i,j \in \alpha\}$ and Cs_α is not finitely based (or not axiomatizable by finitely many schemes, see [8] 4.1.3, 4.1.7) it follows that P_α is not finitely based (or not axiomatizable by finitely many schemes). Since $Crs_\alpha \cap CA_\alpha$ is axiomatizable by a finite scheme (by a result of Resek and Thompson) and $\mathrm{Eq}Cs_\alpha$ is not (by a result of Monk), there is an $\underline{A} \in Crs_\alpha \cap CA_\alpha$ such that $\underline{A} \notin \mathbf{HSP}Cs_\alpha$. But then (iii) gives $\underline{A} \notin \mathbf{SP}P_\alpha = \mathbf{HSP}P_\alpha$ while $\underline{A} \models C_4$. ■(Lemma 4.3)

Now we are in a position to prove Theorem 1.2 (in the Introduction). Recall Definition 1.1 from there.

DEFINITION 4.7 Let

$\mathcal{P}_\mathrm{t} \stackrel{\text{def}}{=} \{\langle \underline{M}, V\rangle \in \mathcal{M}_\mathrm{t} : V$ has the patchwork property and is straightenable $\}$.

$$\stackrel{\mathrm{p}}{\models} \varphi \stackrel{\text{def}}{\iff} (\forall \underline{A} \in \mathcal{P}_\mathrm{t}) \underline{A} \stackrel{\mathrm{m}}{\models} \varphi.$$

■

We note that $\stackrel{\mathrm{p}}{\models}$ is again, like $\stackrel{\mathrm{k}}{\models}$ and $\stackrel{\mathrm{m}}{\models}$, a generalization of $\models$.

COROLLARY 4.1 (i) *$\{\varphi \in \mathrm{F_t} : \stackrel{\mathrm{k}}{\models} \varphi\}$ is decidable, that is, for every formula it is decidable whether it is valid in generalized Kripke-models. Similarly, $\{\varphi \in \mathrm{F_t} : \stackrel{\mathrm{m}}{\models} \varphi\}$ is decidable, but $\{\varphi \in \mathrm{F_t} : \stackrel{\mathrm{p}}{\models} \varphi\} = \{\varphi \in \mathrm{F_t} : \models \varphi\}$ is undecidable.*

(ii) *$\{\varphi \in \mathrm{RF_t} : \models \varphi\}$ and $\{\varphi \in \mathrm{SRF_t} : \not\models \neg\varphi\}$ is decidable, that is, validity is decidable for relativized formulas and satisfiability is decidable for ordinary relativized formulas.*

We will prove Corollary4.1 after Remark4.4.

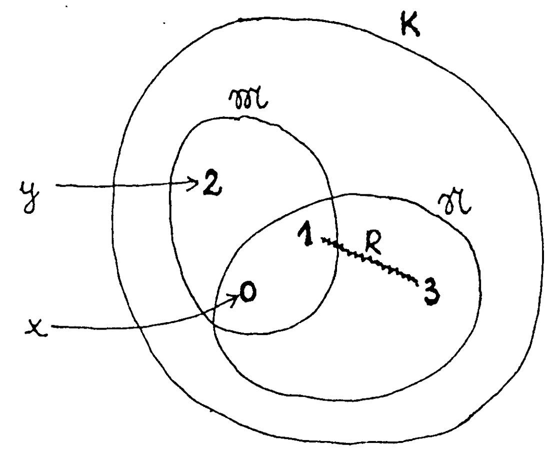


Figure 4.

REMARK 4.4 (i) The definition of $\mathcal{K}_t$ is more general than that of the usual Kripke-models in that we do not require the universes of members of $K \in \mathcal{K}_t$ to be disjoint from each other. The following example shows that $\overset{k}{\not\models} \exists x \exists y \varphi \leftrightarrow \exists y \exists x \varphi$. Let t contain a two-place relation symbol R. Let $K = \{\underline{M}, \underline{N}\}$, where $M = \{0, 1, 2\}$, $N = \{0, 1, 3\}$, $R^{\underline{M}} = 0$ and $R^{\underline{N}} = \{(1, 3)\}$. Let $x = \mathrm{v}_0$, $y = \mathrm{v}_1$ and let $k \in {}^{\omega}M$ be such that $k0 = 0$, $k1 = 2$ and $(\forall i > 1)k(i) = 0$. Then $K \overset{k}{\models} \exists y \exists x R(y, x)[k]$ but $K \overset{k}{\not\models} \exists x \exists y R(y, x)[k]$, see Fig. 4.
(ii) Models with a prescribed set of valuations are not new to logic, *viz.* models of many-sorted logic are such. There, valuations must map variables of sort s to the model's universe of sort s. By Corollary 4.1(i) the important thing in the definition of first-order models is not that we allow all possible valuations but that the set of valuations is straightenable and has the patchwork property.
(iii) $\langle \mathrm{F}_t, \mathcal{P}_t, \overset{\mathrm{P}}{\models} \rangle$ is the logic corresponding to the class of algebras $P_\omega \cap D_\omega$, in the sense of [5] or [8] §5.6. ∎

Proof of Corollary 4.1: First we suppose that F_t contains only restricted formulas. Later we will indicate the modifications needed to cover the case of arbitrary formulas. Let $\varphi \in \mathrm{F}_t$, $n = 1 + \max\{i : \mathrm{v}_i \text{ occurs in } \varphi\}$ and let

$$\chi \stackrel{\text{def}}{=} \forall \mathrm{v}_0 \ldots \mathrm{v}_{n-1} \bigwedge \{\exists \mathrm{v}_i R(\mathrm{v}_0 \ldots \mathrm{v}_{n-1}) \\ \leftrightarrow R(\mathrm{v}_0 \ldots \mathrm{v}_{n-1}) : R \in \underline{R}, i \in n \setminus \mathrm{t}(R), R \text{ occurs in } \varphi\}.$$

Let φ' be the formula obtained from φ by replacing the atomic formulas $R(\mathrm{v}_0 \ldots \mathrm{v}_m)$

(where $R \in \underline{R}$) in φ by $R(\mathbf{v}_0 \ldots \mathbf{v}_{n-1})$. Let $h : \underline{R} \longrightarrow \omega$ be such that $h(R) = n$ for all $R \in \underline{R}$. Then $\bar{\varphi} \stackrel{\text{def}}{=} \chi \rightarrow \varphi' \in \mathrm{F}_h$ and it is not hard to check that $\stackrel{\mathrm{k}}{\models} \varphi \Leftrightarrow \stackrel{\mathrm{k}}{\models} \bar{\varphi}$. In fact, the same holds with $\stackrel{\mathrm{p}}{\models}$ and $\models$ in place of $\stackrel{\mathrm{k}}{\models}$ [8].

Define the function $\tau\mu : \mathrm{F}_h \longrightarrow \mathrm{Tm}_{\underline{R}}(\mathrm{cyl}_\omega)$ as in section 2. Let $K \in \mathcal{K}_h$, $E[K] \stackrel{\text{def}}{=} \{k \restriction n : k \in \mathrm{Val}(K)\}$ and, for all $\psi \in \mathrm{F}_h$, $\tilde{\psi}^K \stackrel{\text{def}}{=} \{k \restriction n : K \stackrel{\mathrm{k}}{\models} \psi[k]\}$. Let $k[K] : \underline{R} \longrightarrow \mathrm{Sb}E[K]$ be such that $k[K]R = (R(\mathbf{v}_0 \ldots \mathbf{v}_{n-1}))^{\tilde{K}}$ for all $R \in \underline{R}$.

It is not hard to check that

$$(*) \qquad K \stackrel{\mathrm{k}}{\models} \bar{\varphi} \Longleftrightarrow \underline{Sb}E[K] \models \tau\mu(\bar{\varphi}) = 1[k[K]]$$

and

$$(**) \qquad \text{if } E \text{ is a } G_n\text{-unit and } k : \underline{R} \longrightarrow \mathrm{Sb}E, \text{ then there is a } K \in \mathcal{K}_h \text{ such that } E = E[K] \text{ and } k = k[K].$$

From $(*)$ and $(**)$ it follows that $\stackrel{\mathrm{k}}{\models} \varphi \Leftrightarrow \stackrel{\mathrm{k}}{\models} \bar{\varphi} \Leftrightarrow G_n \models \tau\mu(\bar{\varphi}) = 1$. The proof of $\stackrel{\mathrm{m}}{\models} \varphi \Leftrightarrow Crs_n \models \tau\mu(\bar{\varphi}) = 1$, $\stackrel{\mathrm{p}}{\models} \varphi \Leftrightarrow P_n \cap D_n \models \tau\mu(\bar{\varphi}) = 1$ and $\models \varphi \Leftrightarrow Cs_n \models \tau\mu(\bar{\varphi}) = 1$ is similar. Now Corollary 4.1(i) for $\stackrel{\mathrm{p}}{\models}$, $\stackrel{\mathrm{k}}{\models}$ with restricted formulas follows from Theorem 4.2 and Lemma 4.3(iii).

Suppose that F_t contains $R(\mathbf{v}_{i_0} \ldots \mathbf{v}_{i_{n-1}})$ for all $R \in \underline{R}$ and $i_0, \ldots, i_{n-1} \in \omega$ if $\mathrm{t}(R) = n$, that is, we allow arbitrary formulas, not just restricted ones. Then, similarly to [8] 4.3.6, one can show that for every $\varphi \in \mathrm{F}_\mathrm{t}$ there is a restricted $\psi \in \mathrm{F}_\mathrm{t}$ such that $\stackrel{\mathrm{k}}{\models} \varphi \leftrightarrow \psi$ and $\stackrel{\mathrm{p}}{\models} \varphi \leftrightarrow \psi$, which implies $\models \varphi \leftrightarrow \psi$. Moreover, ψ can be recursively computed from φ (namely, the recursively given ψ in [8] 4.3.6 will do). With this we have reduced the general case of $\stackrel{\mathrm{k}}{\models}$ and $\stackrel{\mathrm{p}}{\models}$ to the restricted case. This reduction does not work for $\stackrel{\mathrm{m}}{\models}$, so we have to return to the proof of Theorem 4.2.

Let $\varphi \in \mathrm{F}_\mathrm{t}$. Subform$(\varphi)$ denotes the set of subformulas of φ. Let $n \stackrel{\text{def}}{=} 1 + \max\{i : \mathbf{v}_i \text{ occurs in } \varphi\}$. (E, P) is said to be a φ-mosaic if conditions (i), (ii) below hold:

(i) E is a Crs_n-unit and $P : \mathrm{Subform}(\varphi) \longrightarrow \mathrm{Sb}\delta E$.

(ii) (a) $s \in P(R(\mathbf{v}_{i_0} \ldots \mathbf{v}_{i_k})) \Longleftrightarrow z \in P(R(\mathbf{v}_{j_0} \ldots \mathbf{v}_{j_n}))$ if $s, z \in E$, $s_{i_0} = z_{j_0}, \ldots, s_{i_k} = z_{j_k}$ and $R(\mathbf{v}_{i_0} \ldots \mathbf{v}_{i_k}), R(\mathbf{v}_{j_0} \ldots \mathbf{v}_{j_k}) \in \mathrm{Subform}(\varphi)$.

[8] E.g. assume R was originally binary, $n = 4$ and V is straightenable. Let $\langle a, b, c, d\rangle, \langle a, b, e, f\rangle \in V$. If $R(abcd)$, then $R(abbb)$ by $\langle a, b, b, c\rangle, \langle a, b, b, b\rangle \in V$ and by χ. Similarly $R(abbb)$ yields $R(abef)$. Hence the new 4-ary version of R behaves as it was really binary.

(b) $P(\mathsf{v}_i = \mathsf{v}_j) = \mathrm{D}_{ij}^{[E]}$ if $\mathsf{v}_i = \mathsf{v}_j \in \mathrm{Subform}(\varphi)$
$P(\psi \wedge \eta) = P(\psi) \cap P(\eta) \cap E$ if $\psi \wedge \eta \in \mathrm{Subform}(\varphi)$
$P(\neg\psi) = E \setminus P(\psi)$ if $\neg\psi \in \mathrm{Subform}(\varphi)$
$P(\psi) \cap E \subseteq P(\exists \mathsf{v}_i \psi) = \mathrm{C}_i^{[\delta E]} P(\exists \mathsf{v}_i \psi)$ if $\exists \mathsf{v}_i \psi \in \mathrm{Subform}(\varphi)$.

Repeating the proof of Theorem 4.2 with this new notion of mosaic gives a decision procedure for the set $\{\varphi \in \mathrm{F_t} : \overset{\mathrm{m}}{\models} \varphi\}$.

Now we begin the proof of (ii). Let ρ be an atomic formula. Define the function $\mathrm{rl}(\rho) : \mathrm{F_t} \longrightarrow \mathrm{F_t}$ by

$$\mathrm{rl}(\rho)\eta \stackrel{\mathrm{def}}{=} \rho \wedge \eta \text{ if } \eta \text{ is an atomic formula}$$

$$\mathrm{rl}(\rho)(\varphi \wedge \psi) \stackrel{\mathrm{def}}{=} \mathrm{rl}(\rho)\varphi \wedge \mathrm{rl}(\rho)\psi,$$

$$\mathrm{rl}(\rho)(\neg\varphi) \stackrel{\mathrm{def}}{=} \rho \wedge \neg\mathrm{rl}(\rho)\varphi,$$

$$\mathrm{rl}(\rho)(\exists \mathsf{v}_i \varphi) \stackrel{\mathrm{def}}{=} \rho \wedge \exists \mathsf{v}_i \mathrm{rl}(\rho)\varphi.$$

Let ρ be an atomic formula and let $\varphi \in \mathrm{F_t}$ be such that all variable symbols occurring in φ occur in ρ. Suppose first that $\rho = R(\mathsf{v}_0 \dots \mathsf{v}_{n-1})$ is restricted. Then, using the above methods, it is not hard to show that $\not\models \neg\mathrm{rl}(\rho)\varphi \Leftrightarrow \overset{\mathrm{m}}{\not\models} \neg\varphi$, thus satisfiability of $\mathrm{rl}(\rho)\varphi$ is decidable.

Now suppose that $\rho = R(\mathsf{v}_{i_0} \dots \mathsf{v}_{i_{n-1}})$. For all $k < n$ let $f(\mathsf{v}_{i_k}) = v_l$ where $l = \min\{m : \mathsf{v}_{i_m} = \mathsf{v}_{i_k}\}$, and let φ' be the formula obtained from φ by replacing each variable symbol x in φ by $f(x)$. Let

$$R \stackrel{\mathrm{def}}{=} \{(k, l) \in {}^2n : f(\mathsf{v}_{i_k}) = v_l\}$$

and

$$Crs_\alpha^R \stackrel{\mathrm{def}}{=} \{\underline{A} \in Crs_\alpha : (\forall (k, l) \in R)\underline{A} \models \mathrm{d}_{kl} = 1\},$$

for all α. Let $\mathcal{M}'_\mathrm{t} \stackrel{\mathrm{def}}{=} \{\langle \underline{M}, V\rangle \in \mathcal{M}_\mathrm{t} : V \quad \text{is a } Crs_\omega^R\text{-unit}\}$, and for all $\psi \in \mathrm{F_t}$

$$\overset{\mathrm{m}'}{\models} \psi \stackrel{\mathrm{def}}{\Longleftrightarrow} (\forall \langle \underline{M}, V\rangle \in \mathcal{M}'_\mathrm{t})\langle \underline{M}, V\rangle \overset{\mathrm{m}}{\models} \psi.$$

Then, just like before, it is not hard to show that $\{\psi \in \mathrm{F_t} : \overset{\mathrm{m}'}{\models} \psi\}$ is decidable and

$$\not\models \neg\mathrm{rl}(\rho)\varphi \Longleftrightarrow \overset{\mathrm{m}'}{\not\models} \neg\varphi'.$$

So, satisfiability of $\mathrm{rl}(\rho)\varphi$ is decidable. Since every $\psi \in \mathrm{SRF_t}$ is of the form $\mathrm{rl}(\rho)\varphi$ with the above conditions, with this we have shown that satisfiability is decidable for members of $\mathrm{SRF_t}$. Now let φ be relativized. Then φ is of the form $\rho \rightarrow \psi$, where $\psi \in \mathrm{RL}(\rho)$. Thus to decide whether φ is valid it is enough to decide satisfiability of $\rho \wedge \neg\psi$. It can be shown by induction that $\models \rho \wedge \neg\psi \leftrightarrow \mathrm{rl}(\rho)\neg\psi$. Since $\mathrm{rl}(\rho)\neg\psi \in \mathrm{SRF_t}$, its satisfiability is decidable.

By the above we also proved that

$$(***) \qquad \models (\rho \to \psi) \text{ iff } \overset{\mathrm{m}}{\models} \psi.$$

■(Corollary 4.1)

REMARK 4.5 By [17], D_α and Crs_α have the super amalgamation property (SUPAP). Hence by Maksimova [11], the corresponding logics have the strong Craig Interpolation property (and therefore also Beth's definability property). We note that SUPAP implies strong amalgamation, i.e. SAP. In this connection cf. also [12] and [22]. $\mathbf{I}\mathsf{Crs}_\alpha$ is axiomatizable by a schema of equations using one variable, cf. Monk [14], D_α is axiomatizable by a finite schema of equations, cf. Andréka-Thompson [6], and for $\alpha < \omega$, G_α is axiomatizable by a finite set of equations, [24]. By [4], these axiomatizability results provide the corresponding logics with elegant, strongly complete Hilbert style proof calculi.

OPEN PROBLEMS 4.1 (i) *Is* $\mathrm{Eq}(\mathbf{SRl}\mathsf{CA}_\alpha)$ *decidable for* $\alpha > 2$?

(ii) *Let* NA_α *be defined as in [6]. Is* $\mathrm{Eq}(\mathsf{NA}_\alpha)$ *decidable? (We note that* $\mathsf{NA}_\alpha \subseteq \mathsf{NCA}_\alpha$.*)*

<u>Consequences for multi-modal logics, for combining modal logics and for other kinds of algebras:</u>

As in Henkin-Monk-Tarski [8] Part II, Df_α denotes the class of diagonal-free CA_α's. RDf_α denotes the class of representable Df_α's (this class was denoted as Gsdf_α in [8] Part II). As in [8] Part II, $\mathbf{Rl}$ denotes the operator of relativization, e.g. $\mathbf{Rl}\mathsf{Df}_\alpha$ is the class of relativized Df_α's. Below we will use the cylindric equations $C_0 - C_7$ recalled from Henkin-Monk-Tarski [8] at the end of section 2.5 way above.

COROLLARY 4.2 (i) *The varieties* $\mathbf{SRl}\mathsf{Df}_\alpha$ *and* $\mathbf{SRl}\mathsf{RDf}_\alpha$ *have decidable equational theories, i.e.* $\mathrm{Eq}(\mathbf{Rl}\mathsf{Df}_\alpha)$ *and* $\mathrm{Eq}(\mathbf{Rl}\mathsf{RDf}_\alpha)$ *are decidable.*

(ii) *The equations implied by* $C_0 - C_3$ *are decidable, i.e.* $\mathrm{EqMod}(C_0 - C_3)$ *is decidable.*

(iii) $\mathbf{SRl}\mathsf{Df}_\alpha = \mathbf{SRl}\mathsf{RDf}_\alpha$.

Proof: $\mathbf{Rl}\mathsf{Df}_\alpha$ is denoted by Dr_α in [8] Part II. By Thm.5.1.32 of [8] Part II, p.191, $\mathbf{SRl}\mathsf{Df}_\alpha = \mathrm{Mod}(C_0 - C_3)$ is a variety. A theorem of A. Simon (cf. [20]Theorem.8 or equivalently Lambalgen-Simon [10]) says that $\mathbf{SRl}\mathsf{RDf}_\alpha = \mathrm{Mod}(C_0 - C_3)$. Therefore we have

(*) $\quad \mathbf{SRlRDf}_\alpha = \mathrm{Mod}(C_0 - C_3) = \mathbf{SRlDf}_\alpha$.

Claim 4.3 $\mathbf{SRlRDf}_\alpha = \mathbf{ISRd}(\mathsf{Crs}_\alpha)$, *that is* $\mathbf{SRlRDf}_\alpha$ *coincides with the class of subalgebras of diagonal-free (i.e.* d_{ij}*-free) reducts of* $\mathbf{I}\mathsf{Crs}_\alpha$*'s.*

This claim easily follows from the definitions.

By Thm.4.2 way above, $\mathrm{Eq}(\mathsf{Crs}_\alpha)$ is decidable. Now, we show how to decide $\mathrm{Eq}(\mathbf{SRlRDf}_\alpha)$. Let e be an equation in the language of $\mathbf{RlRDf}_\alpha$. Then $\mathbf{RlRDf}_\alpha \models e$ iff $\mathbf{SRlRDf}_\alpha \models e$ iff $\mathsf{Crs}_\alpha \models e$, by Claim4.3 above. Therefore, to decide whether $e \in \mathrm{Eq}(\mathbf{RlRDf}_\alpha)$, it is sufficient to decide whether $e \in \mathrm{Eq}(\mathsf{Crs}_\alpha)$ and by Thm.4.2, the latter is decidable. By (*) above, the same method decides $\mathrm{Eq}(\mathbf{RlDf}_\alpha)$ as well as $\mathrm{EqMod}(C_0 - C_3)$. This proves that all the equational theories in question are decidable. ■(Corollary4.2)

Our algebraic Corollary4.2 above has some logical consequences. In particular, consider the propositional modal logic $S5$. Let α-dimensional $S5$ be the propositional modal logic having α-many unary modalities $\{\Diamond_i : i < \alpha\}$ and the usual axioms of $S5$ postulated for each $\Diamond_i$. We also assume the usual rules, modus ponens and generalization. There are no further axioms or rules.

COROLLARY 4.3 *Let* α *be an arbitrary ordinal. Then* α*-dimensional* $S5$ *is decidable.*

Proof: The algebraic counterpart of α-dimensional $S5$ is $\mathbf{SRlDf}_\alpha$ (this should be straightforward but cf. e.g. [4]). Then we are done by Corollary4.2. ■(Corollary4.3)

COROLLARY 4.4 α*-dimensional* $S5$ *is strongly complete for the "set-theoretic" frames obtainable from* $\mathbf{SRlRDf}_\alpha$ *or equivalently from the greatest elements of* Crs_α*'s.*

Proof: Immediate by Corollary4.2(iii), cf.[4]. ■(Corollary4.4)

COROLLARY 4.5 α*-dimensional* $S5$ *has Craig's Interpolation Property (in its stronger form) and Beth's Definability Property.*

Proof: Again we use the fact that the algebraic counterpart of α-dimensional $S5$ is $\mathbf{SRlRDf}_\alpha$. By [17], $\mathbf{SRlRDf}_\alpha$ has the strong amalgamation property (SAP) and it also has what Maksimova calls super SAP (SUPAP). Cf. also [22], [12]. SAP and SUPAP imply Beth's and Craig's peoperties for the logic in question as is explained e.g. in [4],[22],[12],[8] Part II. ■(Corollary4.5)

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