FREE ALGEBRAS AND DECIDABILITY IN ALGEBRAIC LOGIC

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CONTENTS

Summary of Chapters II. and III.

We use the notation of [HMT] and [NFr]. Difference: $\varphi(\!(u,v)\!)$ denotes here $\varphi(u,v)$ introduced on p.23 of [NFr].

SUMMARY OF CHAPTER II.

MAIN THEOREM 1 $\mathfrak{Fr}_1 CA_3$ is not atomic.

DEFINITION 6 (i) Let $\{u,v,w\}=\{x,y,z\}$ and $i,j\in 2^*$, $k\in 2$. Then

$u_{\langle\rangle}=v_{\langle\rangle} \overset{df}{\Longleftrightarrow} u=v$. In the following, instead of

$u_i=v_{\langle\rangle}$ and $u_{\langle\rangle}=v_j$ we shall simply write $u_i=v$ and $u=v_j$ resp.

$u_k=v \overset{df}{\Longleftrightarrow} v=u_k \overset{df}{\Longleftrightarrow} p_k(\!(u,v)\!)$

$u_{ik}=v \overset{df}{\Longleftrightarrow} v=u_{ik} \overset{df}{\Longleftrightarrow} \exists w(u_i=w, w_k=v)$ if $i\neq\langle\rangle$,

$u_i=v_j \overset{df}{\Longleftrightarrow} \exists w(u_i=w, v_j=w)$ if $i\neq\langle\rangle\neq j$.

$x_i=x_j \overset{df}{\Longleftrightarrow} \exists y(x=y, x_i=y_j)$,

$y_i=y_j \overset{df}{\Longleftrightarrow} \exists x(x=y, x_i=y_j)$,

$z_i=z_j \overset{df}{\Longleftrightarrow} \exists xy(x=y, y=z, x_i=y_j)$.

By the above, the formula "$u_i=v_j$" has been defined for all $i,j\in 2^*$ and $u,v\in\{x,y,z\}$.

(ii) $Ax' \overset{d}{=} \{ (u_i=v_j, v_j=w_k \rightarrow u_i=w_k),\ (u_i=u_i, v_j=v_j \rightarrow \exists w(w_0=u_i, w_1=v_j)) :$
$\{u,v,w\}=\{x,y,z\},\ i,j\in 2^*,\ |i|,|j|,|k|\leq 3 \}$.

Let us note that $Ax'\subseteq Fm_3$ is a <u>finite</u> set of formulas.

$Ax \overset{d}{=} \forall xyz \bigwedge Ax'$. Thus $Ax\in Fm_3^{\Lambda,0}$. ■

REMARK 7 (i) "$u_i=u_i$" means intuitively that "u_i is defined". (ii) Ax is a formulation of π, i.e. $\models \pi\leftrightarrow Ax$. But as we shall see later, $\not\vdash_3 \pi\leftrightarrow Ax$, in particular $\not\vdash_3 \pi\rightarrow Ax$ while $\vdash_3 Ax\rightarrow\pi$. ■

<u>DEFINITION 8</u> Let $\varphi,\psi \in Fm_3^{\Lambda,1}$.

(i) $\varphi u_i \overset{df}{\iff} \exists x(x=u_i, \varphi)$ if $u \in \{y,z\}$ and $i \in 2^*$, and

$pair(x) \overset{d}{=} \exists y p_0(\!(x,y)\!) \wedge \exists y p_1(\!(x,y)\!)$.

(ii) $\varphi \circ \psi \overset{df}{\iff} \exists y(\varphi y_0, \psi y_1, x_0=y_{00}, y_{01}=y_{10}, y_{11}=x_1)$,

$\varphi^{\smile} \overset{df}{\iff} \exists y(\varphi y, y_0=x_1, y_1=x_0)$,

$\varepsilon \overset{df}{\iff} x_0=x_1$, $\dot{1} \overset{df}{\iff} pair(x)$, $\dot{0} \overset{df}{\iff} F$,

$\dot{-}\varphi \overset{df}{\iff} pair(x) \wedge \neg\varphi$, $\varphi+\psi \overset{df}{\iff} \varphi\vee\psi$, $\varphi\cdot\psi \overset{df}{\iff} \varphi\wedge\psi$.

(iii) $Ev^{\Lambda} \overset{d}{=} Ev \overset{d}{=} \{\varphi \in Fm_3^{\Lambda,1} : Ax \vdash_3 \varphi \leftrightarrow \varepsilon \circ \varphi\}$, $\mathcal{E}v^{\Lambda} \in RTA$,

$\mathcal{E}v^{\Lambda} \overset{d}{=} \mathcal{E}v \overset{d}{=} \langle Ev, +, \cdot, \dot{-}, \dot{0}, \dot{1}, \circ, ^{\smile}, \varepsilon \rangle$. ■

<u>THEOREM 9</u> (i) $\mathcal{E}v$ is an algebra, i.e. the set Ev is closed under the operations $\circ$, $^{\smile}$, ε etc.; and $\equiv_{Ax}$ is a congruence on $\mathcal{E}v$. Further, $Ev \supseteq \{\varphi \circ \psi : \varphi,\psi \in Fm_3^{\Lambda,1}\}$.

(ii) $\mathcal{E}v/\!\equiv_{Ax} \in QRA$. ■

<u>REMARK 10</u> (i) We used Ax instead of π because $\mathcal{E}v/\!\equiv_{\pi} \notin RA$. (ii) Remarks on why we used $Ev \subset Fm_3^{\Lambda,1}$ instead of $Fm_3^{\Lambda,1}$. ■

<u>From now on</u>, we allow only binary relation symbols in our languages Λ , $\mathcal{R}$ denotes the set of relation symbols of Λ, and a recursive function $g : Fm_\omega^2 \to RAT$ is fixed such that g satisfies (i),(ii) below (such a g exists by Lemma 3 in Chapter II.1.). Thus from now on, in the definitions p_0, p_1, g are parameters that we do not indicate explicitly.

(i) $g(R(x,y))=R$, $g(\neg\varphi)=-g(\varphi)$, $g(\varphi\wedge\psi)=(g\varphi)\cdot(g\psi)$, $g(\varphi\vee\psi)=(g\varphi)+(g\psi)$ for all relation symbols R and formulas φ,ψ .

(ii) g preserves meaning, i.e. let $\mathfrak{M}$ be a model, $a,b \in M$, $\varphi \in Fm_\omega^2$ and assume that either $\mathfrak{M} \models \pi$ or $\varphi \in Fm_3^2$. Then

$$\mathfrak{M} \models \varphi[a,b] \iff (a,b) \in (g\varphi)^{\mathcal{R}(M)}(\langle R^{\mathfrak{M}} : R \in \mathcal{R}\rangle).$$

DEFINITION 11 (i) Let $\varphi \in Fm_\omega$ be arbitrary. Then

$\varphi(x_0,x_1) \stackrel{d}{=} \exists yz(z=x_0, y=x_1, \varphi(\!(z,y)\!))$.

(ii) The function $\mathbf{h} : RAT \to Fm_3^1$ is defined as follows:

$\mathbf{h}(R) \stackrel{d}{=} R(x_0,x_1)\circ\varepsilon$ for all $R \in \mathcal{R}$, and

$\mathbf{h} : \underset{\sim}{RAT} \to \mathfrak{Ev}$ is a homomorphism, i.e.

$\mathbf{h}(\tau;\sigma) \stackrel{d}{=} \mathbf{h}(\tau)\circ\mathbf{h}(\sigma)$, $\mathbf{h}(\tau^\cup) \stackrel{d}{=} \mathbf{h}(\tau)^\omega$, $\mathbf{h}(1') \stackrel{d}{=} \varepsilon$,

$\mathbf{h}(-\tau) \stackrel{d}{=} pair(x)\wedge\neg\mathbf{h}(\tau)$, $\mathbf{h}(\tau\cdot\sigma) \stackrel{d}{=} \mathbf{h}(\tau)\wedge\mathbf{h}(\sigma)$, $\mathbf{h}(\tau+\sigma) \stackrel{d}{=} \mathbf{h}(\tau)\vee\mathbf{h}(\sigma)$,

$\mathbf{h}(1) \stackrel{d}{=} pair(x)$, $\mathbf{h}(0) \stackrel{d}{=} \mathbf{F}$.

(iii) The functions $\kappa, \bar{\kappa}, \kappa' : Fm_\omega^2 \to Fm_3^1$ are defined as follows:

$\bar{\kappa}\varphi \stackrel{d}{=} \forall x(pair(x) \to \kappa'\varphi)$, $\kappa'\varphi \stackrel{d}{=} \mathbf{hg}\varphi$,

$\kappa\varphi \stackrel{d}{=} \forall x([Ax^* \wedge pair(x)] \to \kappa'\varphi)$, where $Ax^* \stackrel{d}{=} Ax \wedge \bar{\kappa}\pi$. ■

THEOREM 12 (solution of a problem of [TG], p.3.78)

(i) $\pi \models \varphi \iff \pi \vdash_3 \kappa\varphi \iff \vdash_3 \kappa\varphi$, for all $\varphi \in Fm_\omega^2$.

(ii) $\pi \models \varphi \leftrightarrow \kappa\varphi$, for all $\varphi \in Fm_\omega^0$. ■

REMARK 13 Discussion of Thm.12 will follow. ■

THEOREM 14 (solution of Problem 4.14 of [HMT]) Let $1 \leq \beta \leq \omega$ and $3 \leq \alpha < \omega$. Then $\mathfrak{Nr}_\beta CA_\alpha$ is not atomic, further, $\mathfrak{Z}\mathfrak{d}\,\mathfrak{Nr}_\beta CA_\alpha$ is not atomic either. ■

THEOREM 15 (discussion of Thm.9.)

(i) $\pi \not\vdash_3 Ax$ for some $p_0, p_1 \in Fm_3^2$; moreover

(ii) $\mathfrak{Ev}/\equiv_\pi \notin RA$, in particular

$\pi \not\vdash_3 (\varphi\circ\psi)\circ\gamma \leftrightarrow \varphi\circ(\psi\circ\gamma)$ for some $\varphi,\psi,\gamma \in Ev$ and $p_0,p_1 \in Fm_3^2$.

(iii) $\pi \not\models \varphi\circ\varepsilon \to \varphi$, $\pi \not\models \varphi^{\omega\omega} \to \varphi$ for some $\varphi \in Fm_3^1$ and $p_0,p_1 \in Fm_3^2$, where in addition φ is of form $pair(x)\wedge\psi$; and

$\pi \not\vdash_3 \varphi(x_0,x_1)\circ\varepsilon \to \varphi(x_0,x_1)$ for some $p_0,p_1,\varphi \in Fm_3^2$.

(iv) $\widetilde{Ax} \not\vdash_3 (\varphi;\psi);\gamma \leftrightarrow \varphi;(\psi;\gamma)$ for some $\varphi,\psi,\gamma \in Fm_3^2$ and $p_0,p_1 \in Fm_3^2$ where $\varphi;\psi = \exists z(\varphi(\!(x,z)\!)\wedge\psi(\!(z,y)\!))$ is the "usual" composition and $Ax \subseteq \widetilde{Ax}$ is a strong extension of π , see the next Def.16. ■

<u>DEFINITION 16</u> (i) $\mathcal{K} \overset{d}{=} \kappa^{*}Fm^{0}_{\omega}$.

(ii) Let $T \subseteq Fm^{0}_{\omega}$, $\varphi \in Fm^{0}_{\omega}$. Then $T \models_{\pi} \varphi \overset{df}{\Longleftrightarrow} T \cup \{\pi\} \models \varphi$.

(iii) Let H be an arbitrary set. We define

$P_0(H) \overset{d}{=} H$, $P_{n+1}(H) \overset{d}{=} P_n(H) \cup (P_n(H) \times P_n(H))$, $P_\omega(H) \overset{d}{=} \bigcup\{P_n(H) : n \in \omega\}$,

let $U \overset{d}{=} P_\omega(H)$, then

$pj_0^H \overset{d}{=} \{(a,b) \in {}^2U : (\exists c \in U) a = (b,c)\}$ and

$pj_1^H \overset{d}{=} \{(a,b) \in {}^2U : (\exists c \in U) a = (c,b)\}$, i.e. pj_0^H and pj_1^H are the usual*/ set theoretic projection functions on $P_\omega(H)$.

Let $\mathfrak{h}$ denote the similarity type containing two binary relation symbols p and q. Then

$\mathfrak{S}(H) \overset{d}{=} \langle P_\omega(H), pj_0^H, pj_1^H \rangle \in Mod(\mathfrak{h})$.

We call $\mathfrak{S}(H)$ a standard model of projection functions.

(iv) Let $\overline{Ax'}$ be the "standard theory" of the projection functions p,q, i.e. let $\overline{Ax'}$ be the set of all first-order formulas with 2 variables that are true in $\mathfrak{S}(\omega)$. I.e.

$\overline{Ax'} \overset{d}{=} \{\varphi \in Fm_3^{\Lambda,0} : \mathfrak{S}(\omega) \models \varphi\}$.

Let $p_0, p_1 \in Fm_3^2$ and $\varphi \in Fm_3^{\Lambda,0}$. Then $\varphi(p_0,p_1) \in Fm_3^0$ denotes the formula we obtain from φ by replacing $p(x,y), q(x,y)$ in it everywhere with p_0, p_1 . Now

$\overline{Ax} \overset{d}{=} \overline{Ax}(p_0,p_1) \overset{d}{=} \{\varphi(p_0,p_1) : \varphi \in \overline{Ax'}\}$, $\overline{Ax^{*}} \overset{d}{=} \overline{Ax} \cup \{\overline{\kappa}\pi\}$.

Thus $\overline{Ax}$, $\overline{Ax^{*}} \in Fm_3^0$.

(v) $\mathfrak{R}_H$ denotes the relation algebra generated by pj_0^H , pj_1^H i.e. $\mathfrak{R}_H \overset{d}{=} \mathfrak{Sg}^{(\mathfrak{R})}\{pj_0^H, pj_1^H\}$, where $\mathfrak{R} = \mathcal{R}(P_\omega(H))$.

(vi) We say that $\mathfrak{A} \in CA_3$ is strongly quasi-projective, in symbols $\mathfrak{A} \in \overline{Q}CA_3$, if $\mathfrak{Ra}\,\mathfrak{A}$ contains a subalgebra isomorphic to $\mathfrak{R}_\omega$.

${}_1\overline{Q}CA_3 \overset{d}{=} \{\mathfrak{A} \in CA_3 : \mathfrak{R}_\omega \in \mathbf{IS}\{\mathfrak{Ra}\,\mathfrak{A}\}$ and $(\exists e \in Nr_2\mathfrak{A}) A = Sg\{e\}\}$. ■

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*/ We shall use this notation when H will not contain any pairs.

<u>THEOREM 17</u> (discussion of Thm.12)

(i) $\pi \models \varphi$ and $\pi \not\vdash_3 \bar{\kappa}\varphi$ for some $\varphi \in Fm_3^2$ and "good" p_0, p_1, g.

(ii) $\pi \not\vdash_3 \varphi \leftrightarrow \kappa\varphi$ for some $\varphi \in Fm_3^0$ and "good" p_0, p_1, g ; moreover

(iii) $\overline{Ax^*} \not\vdash_3 \varphi \rightarrow \kappa\varphi$ for some $\varphi \in Fm_3^0$ and "good" p_0, p_1, g ; and
$\overline{Ax^*} \not\vdash_3 \kappa\varphi \rightarrow \varphi$ for some $\varphi \in Fm_3^0$ and "good" p_0, p_1, g.

(iv) $T \models_\pi \varphi \iff T \vdash_3 \varphi$, for all $T \subseteq \mathcal{K} \ni \varphi$.

(v) $T \models \varphi \not\Longrightarrow T \vdash_3 \varphi$ for some $\overline{Ax^*} \subseteq T \subseteq Fm_3^0$, $\varphi \in \kappa^* Fm_3^0$ and "good" p_0, p_1, g.

(vi) $T \models_\pi \varphi \iff \kappa^* T \vdash_3 \kappa\varphi$, for all $T \subseteq Fm_\omega^0 \ni \varphi$.

(vii) $T \models \varphi \not\Longrightarrow T \vdash_3 \kappa\varphi$ and $T \models \kappa\psi \not\Longrightarrow T \vdash_3 \psi$, for some
$\overline{Ax^*} \subseteq T \subseteq Fm_3^0 \ni \varphi, \psi$ and "good" p_0, p_1, g.

(viii) ${}_1\bar{Q}CA_3 \nsubseteq RCA_3$. ■

<u>REMARK 18</u> (i) It can be proved (similarly to Thm.14) that $\mathfrak{Fr}_\beta RA$ and $\mathfrak{Fr}_\beta SA$ are not atomic. Moreover, $\mathfrak{Fr}_\beta K$ is not atomic if $K \subseteq SA$, $\mathfrak{R}(U) \in K$ for some infinite U and the equations valid in K are recursively enumerable. We do not know whether $\mathfrak{Fr}_\beta NA$, $\mathfrak{Fr}_\beta WA$ or $\mathfrak{Fr}_\beta Crs_\alpha$ $(2<\alpha<\omega$, $0<\beta<\omega)$ are atomic or not.

(ii) We can prove $\bar{Q}SA \nsubseteq RA$ exactly as $\bar{Q}CA_3 \nsubseteq RCA_3$.

(iii) We sketch a direct proof of Thm.12 that does not use Tarski's representation theorem $QRA \subseteq RRA$.

(iv) (about a logic $\mathcal{L}_3$ used in [TG] that is stronger than our L_3)

(v) There is a recursive $G : Fm_\omega^0 \rightarrow RAT$ such that $[\pi \models \varphi \iff SA \models G\varphi = 1$, for all $\varphi \in Fm_\omega^0]$. ■

<u>THEOREM 19</u> (solution of Problem 2.7 of [HMT]) There is $b \in Fr_1 CA_3$ that generates $\mathfrak{Fr}_1 CA_3$ but not freely. In general, $\mathfrak{Fr}_\beta CA_\alpha$ has a β-element generating system that does not generate it freely, if $0<\beta$ and $3 \leq \alpha$. This generating system is irredundant. ■

<u>REMARK 20</u> (i) Thm.19 remains true for $\mathfrak{Fr}_\beta K$ if $K = RA$ or if $\mathfrak{R}(\omega) \in K \subseteq SA$, but it fails for $K = WA$ or $K = NA$. (ii) (about the proof of Thm.19) ■

SUMMARY OF CHAPTER III.

EqK denotes the set of equations valid in K.

COROLLARY 1 (Maddux) $EqCA_3$ is undecidable. ■

DEFINITION 2 Let K be a class of structures (in particular, K may be a class of algebras).

(i) K is said to be strongly decidable if its similarity type is finite and if there is a recursive function $f : \omega \to \omega$ such that

a) $(\forall \mathfrak{A} \in K)(\forall X \subseteq_\omega A)(\exists \mathfrak{B} \in K)[X \upharpoonright \mathfrak{A} = X \upharpoonright \mathfrak{B}$ and $|B| \leq f(|X|)]$, and

b) $\{\mathfrak{A} \in K : A \in \omega\}$ is decidable.

(ii) $\mathbf{F}K \overset{d}{=} \{\mathfrak{A} \in K : |A| < \omega\}$.

(iii) QeqK denotes the set of quasi-identities valid in K. ■

REMARK 3 Let K be a class of algebras. (i) If K is strongly decidable then EqK is decidable, moreover the set of universal formulas valid in K is decidable. In particular, QeqK is decidable, hence the word-problem for K is solvable. Further, EqK = Eq$\mathbf{F}$K, moreover $\mathbf{F}$K and K cannot be distinguished by universal formulas.

(ii) Connections between decidability of EqK and EqK=Eq$\mathbf{F}$K.

(iii) CA_2, RCA_2 and WA, NA are all strongly decidable. ■

Chapter III.1.

DEFINITION 4 $NCA_\alpha \overset{d}{=} \{\mathfrak{A} \in CTA_\alpha : \mathfrak{A} \models \{C_0, C_1, C_2, C_3, C_5, C_6, C_7\}\}$. ■

THEOREM 5 Let $\alpha \leq \omega$.

(i) $EqNCA_\alpha$ is decidable.

(ii) $EqNCA_\alpha = Eq\mathbf{F}NCA_\alpha$, if $\alpha < \omega$.

(iii) $QeqNCA_\alpha \neq Qeq\mathbf{F}NCA_\alpha$, if $\alpha \geq 3$. Thus NCA_α is not strongly decidable if $\alpha \geq 3$. ■

REMARK 6 (i) The condition $\alpha < \omega$ cannot be omitted in Thm.5(ii) because $EqNCA_\alpha \neq Eq\mathbf{F}NCA_\alpha$ if $\alpha \geq \omega$. We do not know whether the word-problem is solvable for NCA_α, $\alpha \geq 3$. NCA_α is strongly decidable for $\alpha < 2$. Let $\alpha \geq 3$. Moreover, if $i \in 8$ then let

$NCA_\alpha^{-i} \stackrel{d}{=} \{\mathfrak{A} \in CTA_\alpha : (\forall j \in 8 \sim \{4,i\})\, \mathfrak{A} \models C_j\}$. Now, if $i \in 8$, $i \neq 0$ then $[NCA_\alpha^{-i}$ is strongly decidable iff $i \in \{2,3,7\}]$.

We give the constructions showing that WA and NA are strongly decidable.

(ii) About the proof of Thm.5. ■

COROLLARY 7 If we replace ⟪4⟫ in the proof system $\vdash_\alpha$ with the following ⟪4b⟫ - ⟪4d⟫ then we obtain a proof system where the set of provable formulas is decidable. If we replace ⟪4⟫ in $\vdash_\alpha$ with ⟪4a⟫ - ⟪4d⟫ then the set of provable formulas remains unchanged.

⟪4a⟫ $\forall v_i \forall v_j \varphi \rightarrow \forall v_j \forall v_i \varphi$

⟪4b⟫ $\forall v_k \varphi \rightarrow \forall v_k \forall v_k \varphi$

⟪4c⟫ $\exists v_k \varphi \rightarrow \forall v_k \exists v_k \varphi$

⟪4d⟫ $R(\bar{x}) \rightarrow \forall v_k R(\bar{x})$ if $v_k \notin Rg\bar{x}$, and $R(\bar{x})$ is an atomic formula. ■

Chapter III.2.

THEOREM 8 (recalled without proof) Let α be arbitrary.

(i) $ICrs_\alpha$ is a variety, i.e. it is axiomatizable with identities.

(ii) $ICrs_\alpha$ is not finitely axiomatizable.

(iii) $ICrs_\alpha$ is not axiomatizable with finitely many schemes, but it is axiomatizable with countably many schemes. ■

MAIN THEOREM 2 $EqCrs_\alpha$ is decidable for all $\alpha \leq \omega$. ■

DEFINITION 9 Let α be arbitrary.

$D_\alpha \stackrel{d}{=} \{\mathfrak{A} \in Crs_\alpha : (\forall s \in 1^{\mathfrak{A}})(\forall i,j \in \alpha) s(i/s_j) \in 1^{\mathfrak{A}}\}$,

$G_\alpha \stackrel{d}{=} \{\mathfrak{A} \in Crs_\alpha : (\forall s \in 1^{\mathfrak{A}})\ {}^{\alpha}(Rgs) \subseteq 1^{\mathfrak{A}}\}$.

Let $K \subseteq Crs_\alpha$. We say that V is a K-unit if $\mathfrak{S}\mathfrak{b}\, V \in K$. ■

THEOREM 10 (i) EqG_α and $EqCrs_\alpha$ are decidable for all $\alpha \leq \omega$.

(ii) EqD_α is decidable for $\alpha < \omega$.

(iii) If $\alpha < \omega$ and $K \subseteq Crs_\alpha$ satisfies conditions (a)-(d) below then EqK is decidable.

(a) Union of K-units is a K-unit, i.e.
$(\forall V \in \mathcal{V})\ \mathfrak{Gb}V \in K \implies \mathfrak{Gb}(\bigcup \mathcal{V}) \in K.$

(b) Restriction of a K-unit is a K-unit, i.e.
$\mathfrak{Gb}V \in K \implies \mathfrak{Gb}(V \cap {}^{\alpha}H) \in K.$

(c) Base-isomorphic image of a K-unit is a K-unit, i.e.
$\mathfrak{Gb}V \in K$ and $f : \mathrm{base}(V) \twoheadrightarrow U \implies \mathfrak{Gb}\{f \circ s : s \in V\} \in K.$

(d) $\mathfrak{A} \in K \implies \mathfrak{Gb}1^{\mathfrak{A}} \in K.$ ■

<u>LEMMA 10.10.</u> Let $\gamma \subset \alpha$, $2 \le |\gamma| < \omega$ and $\tau \in Tm(cyl_\gamma)$.

(i) $\mathbf{Rd}_\gamma \mathbf{I}Crs_\alpha = \mathbf{I}Crs_\gamma$ and
$Crs_\alpha \models \tau = 1 \iff Crs_\gamma \models \tau = 1.$

(ii) $\mathbf{HSP}\ \mathbf{Rd}_\gamma G_\alpha \subset \mathbf{I}G_\gamma$, but
$G_\alpha \models \tau = 1 \iff G_\gamma \models \tau = 1$, if $ind(\tau) \subset \gamma$.

(iii) $\mathbf{HSP}\ \mathbf{Rd}_\gamma D_\alpha \subset \mathbf{I}D_\gamma$, and for every $n \le |\gamma| - 2$ there is $\sigma \in Tm(cyl_\gamma)$ such that
$D_\alpha \models \sigma = 1 \not\Rightarrow D_\gamma \models \sigma = 1$ and $|\gamma \sim ind(\sigma)| \ge n$. Further,
$D_\alpha \models \delta = 1 \not\Rightarrow G_\gamma \models \delta = 1$ for some $\delta \in Tm(cyl_\gamma)$. ■

We note that the condition "$ind(\tau) \subset \gamma$" cannot be omitted in L.10.10(ii) by the last statement of L.10.10(iii).

<u>THEOREM 11</u> (i) $EqG_\alpha \ne EqD_\alpha$ if $\alpha \ge 2$.
(ii) $EqD_\alpha \ne Eq\mathbf{P}D_\alpha$ if $\alpha \ge \omega$. ■

<u>REMARK 12</u> (i) Probably, the following can be proved with the methods of L.10.10: $\mathbf{ISRd}_\alpha G_{\alpha+1} = \mathbf{ISRd}_\alpha G_{\alpha+\beta}$ for all $\beta \ge 1$, but $\mathbf{ISRd}_\alpha D_{\alpha+n} \ne \mathbf{ISRd}_\alpha D_{\alpha+n+1}$ for all $n \in \omega$.

(ii) Crs_α is strongly decidable for $\alpha \le 2$, and probably so are G_α and D_α (for $\alpha \le 2$). Let $\alpha \ge 3$. We do not know whether $EqCrs_\alpha = Eq\mathbf{P}Crs_\alpha$ or whether the word problem is solvable for Crs_α. We do not know whether EqD_ω is decidable or not. ■

<u>DEFINITION 13</u> We say that V has the <u>patchwork-property</u> if
$(\forall s, z \in V)(\forall H \subseteq \alpha)[(H \upharpoonright s) \cup (\alpha \sim H) \upharpoonright z] \in V.$

$P_\alpha \stackrel{d}{=} \{\mathfrak{A} \in Crs_\alpha : 1^{\mathfrak{A}}$ has the patchwork-property$\}$. ■

LEMMA 14 (i) $[\mathfrak{Gb}V \models C_4 \iff \mathfrak{Gb}V \in \mathbf{P}P_\alpha]$, for any Crs_α-unit V. But $\mathfrak{A} \models C_4$ for some $\mathfrak{A} \in Crs_\alpha \sim \mathbf{HSP}P_\alpha$ if $\alpha \geq 3$.

(ii) $\mathbf{HSP}\ P_\alpha = \mathbf{SP}\ P_\alpha = \mathbf{I}\{\mathfrak{A} \in Crs_\alpha : 1^{\mathfrak{A}}$ is a union of P_α-units with disjoint bases$\}$, and

EqP_α is undecidable (and is not finitely axiomatizable) if $\alpha \geq 3$.

(iii) $\mathbf{SP}(P_\alpha \cap D_\alpha) = \mathbf{SP}\ Cs_\alpha = RCA_\alpha$. ■

DEFINITION 15

(i) Let $K \subseteq Mod_t$. We say that K is a generalized Kripke-model or that K is a partial model, in symbols $K \in \mathcal{K}_t$, if

$(\forall \mathfrak{M}, \mathfrak{N} \in K)\ (M \cap N)\rceil\mathfrak{M} = (M \cap N)\rceil\mathfrak{N}$.

We define validity of usual first-order formulas in elements of $\mathcal{K}_t$ the natural way. $K \models^k \varphi$ denotes that φ is valid in K.

$\models^k \varphi \quad \overset{df}{\iff} \quad (\forall K \in \mathcal{K}_t)\ K \models^k \varphi$.

(ii) Models with prescribed evaluations of the variables. Let

$\mathcal{M}_t \overset{d}{=} \{\langle \mathfrak{M}, V\rangle : \mathfrak{M} \in Mod_t$ and $V \subseteq {}^\omega M\}$.

We define validity of the usual first-order formulas in elements of $\mathcal{M}_t$ the natural way. $\mathfrak{M} \models^m \varphi$ denotes that φ is valid in $\mathfrak{M}$.

$\models^m \varphi \quad \overset{df}{\iff} \quad (\forall \mathfrak{M} \in \mathcal{M}_t)\ \mathfrak{M} \models^m \varphi$. Let

$\mathcal{P}_t \overset{d}{=} \{\langle \mathfrak{M}, V\rangle \in \mathcal{M}_t : V$ is a $P_\alpha \cap D_\alpha$-unit$\}$.

$\models^p \varphi \quad \overset{df}{\iff} \quad (\forall \mathfrak{M} \in \mathcal{P}_t)\ \mathfrak{M} \models^m \varphi$.

(iii) We say that φ is relativized, in symbols $\varphi \in RF_t$, if φ is of form $\varrho \to \psi$ for some atomic formula ϱ, where ψ is built up from atomic formulas by means of $\neg$, $\wedge$ and "$\exists v_i(\varrho \wedge \ldots)$", and further every variable occurring in ψ occurs in ϱ. We say that φ is relativized in the usual sense, in symbols $\varphi \in SRF_t$, if there is an atomic formula ϱ such that φ is built up from $\{\varrho \wedge \eta : \eta$ is an atomic formula$\}$ by means of $\wedge$, $\varrho\wedge\neg$, $\varrho\wedge\exists v_i$; and moreover all variables occurring in φ occur in ϱ. ■

COROLLARY 16

(i) It is decidable whether a formula is valid in the generalized Kripke-models, i.e.

$\{\varphi \in F_t \ : \ \models^{k} \varphi\}$ is decidable. Similarly,

$\{\varphi \in F_t \ : \ \models^{m} \varphi\}$ is decidable, but the same formulas are valid in the usual and in the "patchwork-models", i.e.

$\{\varphi \in F_t \ : \ \models^{p} \varphi\} = \{\varphi \in F_t \ : \ \models \varphi\}$ is undecidable.

(ii) Validity of relativized formulas and satisfiability of formulas relativized in the usual sense is decidable, i.e. $\{\varphi \in RF_t \ : \models \varphi\}$ and $\{\varphi \in SRF_t \ : \ \not\models \neg\varphi\}$ are decidable. ■

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