LOGIC WITH THREE VARIABLES HAS GÖDEL'S INCOMPLETENESS PROPERTY - THUS FREE CYLINDRIC ALGEBRAS ARE NOT ATOMIC

Németi, I. 1985.

CONTENTS

ABSTRACT We show that the β-generated free CA_α is not atomic if $\beta \geqslant 1$ and $\alpha \geqslant 3$. This is a solution of Problem 4.14 in [HMT]. The heart of the proof is the definition of a reduct of CA_3's which is a (representable) relation algebra. This way we give a positive solution to Tarski's problem [TG] section 3.10, p.3.78. (The solution says, roughly, that full first-order logic, hence set theory, can be interpreted in the equational theory of CA_3 and not only in that of RA. This does not generalize much further.) By showing that not every quasi-projective semi-associative relation algebra is representable, we provide a negative answer to an algebraic part of the same problem of Tarski (see the references to Maddux's work above the formulation of the problem in [TG]), too. We investigate free relation algebras, too, and investigate the logical aspects.

INTRODUCTION

Cylindric and relation algebras are algebraizations of first-order logic. The structures of free cylindric and free relation algebras are quite rich since these are able to recapture the whole of first-order logic, in a sense. One of the first things to investigate about these free algebras is whether they are atomic or not.

$\mathfrak{Fr}_\beta CA_\alpha$ denotes the β-generated free cylindric algebra of dimension α, where $\beta > 0$. The following have already been known: If $\beta \geqslant \omega$ then $\mathfrak{Fr}_\beta CA_\alpha$ is atomless (Pigozzi, [HMT]2.5.13). Assume $0 < \beta < \omega$. If $\alpha < 2$ then $\mathfrak{Fr}_\beta CA_\alpha$ is finite ([HMT]2.5.3(i)), hence atomic. $\mathfrak{Fr}_\beta CA_2$ is infinite but still atomic (Henkin, [HMT]2.5.3(ii),2.5.7(ii)). If $3 \leqslant \alpha < \omega$ then $\mathfrak{Fr}_\beta CA_\alpha$ has infinitely many atoms (Tarski, [HMT]2.5.9), and it was asked in [HMT]as Problem 4.14 whether it is atomic or not. $\mathfrak{Fr}_\beta CA_\alpha$ has exactly 2^β zero-dimensional atoms[*/] (Pigozzi, [HMT]2.5.11). It was conjectured that these are all the atoms if $\alpha \geqslant \omega$ (see [HMT]2.5.12, Problem 2.6).

Here we prove, as a solution of Problem 4.14 in [HMT] Part II p.180 that $\mathfrak{Fr}_\beta CA_\alpha$ is not atomic for $\beta < \omega$ and $\alpha \geqslant 3$. Here we present a metalogical proof, using Gödel's incompleteness theorem for usual first-order logic. This way we prove that $\mathfrak{Zd}\,\mathfrak{Fr}_\beta CA_\alpha$ is not atomic, either, if $\alpha < \omega$. In [N84a], by characterizing the locally finite part of $\mathfrak{Fr}_\beta CA_\alpha$ and this way solving Problem 2.10 of [HMT], we show that $\mathfrak{Zd}\,\mathfrak{Fr}_\beta CA_\alpha$ is atomic if $\alpha \geqslant \omega > \beta$. [HMT] also raised the problem of finding purely algebraic proofs for these properties of free algebras. We have direct, purely algebraic proofs showing that $\mathfrak{Fr}_\beta CA_\alpha$ is not atomic, for $\alpha \geqslant 4$. However, those proofs do not work for $\alpha = 3$ (we have counterexamples in which the crucial lemmas fail), and they are longer than the present metalogical proof. On the other hand, those algebraic proofs show that there is an atom of $\mathfrak{Zd}\,\mathfrak{Fr}_\beta CA_\alpha$ (for $0 < \beta < \omega$ and $4 \leq \alpha < \omega$) below which there is no atom of $\mathfrak{Fr}_\beta CA_\alpha$. We do not know whether this holds for $\alpha = 3$ or not. The algebraic proofs can be found in [N84]. As for the conjecture in [HMT] about the nonzero-dimensional atoms in the case $\alpha \geqslant \omega$, in [N84] we prove that it is true for the free representable CA_α ($\alpha \geqslant \omega$), and we have some partial

---

[*/] There may be much more atoms in $\mathfrak{Zd}\,\mathfrak{Fr}_\beta CA_\alpha$. I.e. the atoms of $\mathfrak{Zd}\,\mathfrak{Fr}_\beta CA_\alpha$ usually are not atoms in $\mathfrak{Fr}_\beta CA_\alpha$.

results that might point into the opposite direction for the free CA_α. Namely, in [N84] we show that there is a nonzero element in $\mathfrak{Fr}_\beta CA_\alpha$ which is below $-d_{ij}$ for all $i,j\in\alpha\sim 2$. This cannot happen in the representable case.

We investigate free relation algebras, too. They are not atomic, either. Actually, we prove more. Namely: SA denotes the variety of semi-associative relation algebras introduced in Maddux [Ma78],[Ma82]. SA is obtained by restricting the associative law from $x;y;z$ to $x;1;1$. We prove that no recursively enumerable variety of SA's (containing at least one full infinite set algebra) has an atomic free algebra. In Németi[N85c] it is shown that this result does not generalize to the broader classes WA and NA introduced in the same works of Maddux. (WA and NA are obtained from SA by further weakening the associativity of relation composition. We note that $RA\subseteq SA\subseteq WA\subseteq NA$.)

A separate section (§1) is devoted to the discussion of connections with logic.: There we show that atomicity of the free CA's correspond to failure of Gödel's incompleteness theorem for the corresponding logics with finitely many variables. Thus the result that $\mathfrak{Fr}_1 CA_3$ is not atomic shows that the logic with three variables, though rather weak*/, is strong enough to have Gödel's incompleteness.

—— . ——

<u>About the method of the proof.:</u> Using the pairing functions technique of Tarski together with Gödel's incompleteness

---

*/ E.g. the fact that $f\circ g$ is a function if f and g are functions is <u>not</u> provable in this logic.

theorem for first-order logic, it is not too difficult to show that $\mathfrak{Fr}_\beta \mathrm{Gs}_\alpha$ is not atomic for $\beta \geqslant 1$ and $\alpha \geqslant 3$, and that the semantical version of Gödel's incompleteness holds for logics with three variables (indeed, these are corollaries of our lemmas 2.2, 2.7 in §2). Using in addition to these Tarski's representation theorem of quasi-projective relation algebras (QRA's), one can show that $\mathfrak{Fr}_\beta \mathrm{CA}_\alpha$, $\mathfrak{Fr}_\beta \mathrm{SNr}_3 \mathrm{CA}_\alpha$, $\mathfrak{Fr}_\beta \mathrm{RA}$, $\mathfrak{Fr}_\beta \mathrm{RRA}$ are not atomic if $\beta \geqslant 1$ and $\alpha \geqslant 4$ (and that the stronger, syntactical version of Gödel's incompleteness holds for first-order logics using $\geqslant 4$ variables). These are corollaries of Lemmas 2.2,3,6,7 of §2. The case $\alpha = 3$ is much more difficult, as often is in cylindric algebra theory (and in finite variable logic).

In §1 we give examples to show that first-order logic with three variables is indeed quite weak. Let L_3 denote first-order logic with 3 variables. Let Fm_3 denote the set of formulas of L_3 and let $\vdash_3$ denote the provability relation of L_3. (A precise definition of $\vdash_3$, and hence of L_3, is in §1. Our L_3 was called restricted three-variable logic in [HMT]§4.3.) Now L_3 is not complete (neither is L_α for $3 \leq \alpha < \omega$), i.e. there are many valid but unprovable formulas in L_3. The heart of our proof for the case $\alpha = 3$ is the definition of a translation function κ and a finite set $Ax \subseteq Fm_3$ of axioms such that

$$(*) \quad (\forall \varphi \in Fm_3) \Big[\, Ax \models \varphi \qquad \text{iff} \qquad Ax \vdash_3 \kappa\varphi \, \Big],$$

thus achieving a kind of completeness for L_3 (cf. Prop.3.3 in §3). We shall call the above (*) a quasi-completeness property (of L_3). This (*) is analogous to Tarski's translation mapping theorem (TMT), see [TG]Thm.4.4(xxxiv) on p.4.47. One can use the above (*) to show that whole first-order logic can be built up in L_3, in spite of the fact that L_3 is very weak. For a stronger version ($\mathcal{L}_3$) of L_3 this is done in [TG], but on the expense of adding a strong scheme of formulas as an axiom scheme to L_3, namely the axiom scheme of associativity of relation composition, (and introducing a strong substitution rule called general Leibniz law). Tarski raised the problem that if one does not add the above scheme of axioms to L_3, then L_3 might remain too weak, i.e. one perhaps cannot build up full first-order logic in L_3. By proving (*) therefore we solved Tarski's problem positively. For a formulation of the problem see [TG]p.3.78, which is in §3.10 of [TG] immediately below item (BIV′) but above Thm. 3.10(i). The history of this problem goes back quite some time (actually, Maddux's work on SA's was motivated by Tarski's asking this problem in the early seventies). Namely: Taking up the extensive work summarized in Schröder[S85], Tarski[T41] started to investigate the connections between the axiom system of relation algebras (which is roughly the same as the above outlined L_3 augmented with the associativity scheme) and first-order logic. He found that all the relation algebra (RA) axioms are provable in L_3 (strengthened with the general Leibniz law) except the associativity scheme. So he raised the problem "how much of RA theory can be carried through in L_3 ?" In Tarski[T53],[T53a] he proved basically our (*) above for "RA-logic" that is basically for L_4. This together with his earlier problem mentioned above gave rise to the problem, roughly speaking, whether (*) holds for L_3. By subsequent developments and partial solutions (e.g. Maddux's discovering of SA's), the problem became richer, obtaining finally the form in which it appears in the monograph [TG]. For this richer problem, we shall see that there is a negative answer, too.

The present results seem to solve another problem in [TG]. Namely, since we have the quasi-completeness property (*) for our L_3 which is equivalent to CA_3, the main objective (formalization of full set theory) of [TG] can be carried through in our L_3 (i.e. in the equational theory of CA_3). This means that despite of the conjecture formulated on p.3.37 (at the beginning of Sec.3.7) of [TG] the main aims of [TG] can be carried through in Tarski's original version of $\mathcal{L}_3$ (that is, in $\mathcal{L}_3$ as defined at the beginning of §3.7 of [TG]) instead of the stronger version of $\mathcal{L}_3$ defined in §3.8 therein: not only associativity called (AX) there but also the general Leibniz law called (AIX') can be avoided. Introducing this general Leibniz law (AIX') had some undesired effects (this is pointed out in [TG]p.3.42_{7-6}, p.3.74^{6-17}, p.3.76^{4-7}) one of them being that it does not fit into the process of algebraization.

The algebras corresponding to L_3 without the associative scheme are Maddux's semi-associative relation algebras, SA's. Thus our (*) above implies that every first-order theory can be represented as an SA (as a positive solution of Tarski's problem). In more detail, all finite first-order theories can be represented as elements of the free SA. And the same holds if SA is replaced with CA_3. From TMT, Tarski proved that every quasi-projective relation algebra (QRA) is representable. (This is the algebraic form of the "quasi" completeness theorem.) Therefore one might think that from (*) one could prove that every quasi-projective SA (QSA) is representable (since this is what Tarski did from TMT for RA's). However, this is not true even under quite strong assumptions (see Thm.3.7 in §3). This gives a negative solution to (a part of) Tarski's above-mentioned problem. (What one can prove from (*) is that a certain generalized reduct of any QSA is representable.) As a corollary, there is a CA_3 $\mathfrak{L}$ with a pair of quasi-projections (moreover, $\mathfrak{Ra}\mathfrak{L}\in QSA$) such that $\mathfrak{L}$ is not representable.

We note that replacing the associativity scheme, which is an infinite set of formulas, with the finite set Ax of formulas is crucial in being able to prove non-atomicity of free algebras (i.e. establishing Gödel's incompleteness property for L_3).

The contents of this paper were presented at the 1985 January Oberwolfach meeting, see [N85].

Acknowledgements We are deeply grateful to J. Donald Monk for calling our attention to the problem of atomicity of free cylindric algebras (in 1983) and to Steve Givant for calling our attention to the "pairing function" technique of Tarski (in 1984). We thank Roger Maddux for teaching us relation algebra theory. Thanks are due to L. Csirmaz, G. Hansoul, R. Maddux and M.M. Richter for many valuable discussions concerning the topic of the present work.

§1. LOGICAL FORMULATION AND INTRODUCTION

1.1. OUR LANGUAGES

First-order logics using finitely many variables have already been widely investigated, see e.g. Henkin[H67],[H73],[H83], [Johnson 73], Maddux[Ma83], Monk[M71], [Poizat 82].
Let α be any ordinal. First we introduce our first-order logic (with equality) using α variables. (For $\alpha \geqslant \omega$ this will be the same as our "normal" first-order logic.) We recall the following from [HMT]§4.3.

Our set of variables is $\{v_i : i\in\alpha\}$. Let β be any ordinal. Let $R = \langle R_i : i\in\beta\rangle$ be the sequence of the relation symbols and let $\varrho = \langle \varrho_i : i\in\beta\rangle$ be the sequence of their arities (in other words, ranks), such that $(\forall i\in\beta)$ $(\varrho_i \leq \alpha , \varrho_i \in \omega)$*/. Let $\Lambda \overset{d}{=} \langle \alpha, R, \varrho\rangle$. Then we say that Λ is our language.

The set Fm^Λ of formulas of Λ is defined the following way. Fm^Λ is the smallest set such that

(i) $\{R_i(v_0,v_1,\dots,v_{\varrho_i-1}) : i\in\beta\} \cup \{v_i=v_j : i,j\in\alpha\} \cup \{\underline{T},\underline{F}\} \subseteq$ $\subseteq Fm^\Lambda$

(ii) $\{\exists v_i\varphi , \varphi\vee\psi , \varphi\wedge\psi , \neg\varphi\} \subseteq Fm^\Lambda$ whenever $i\in\alpha$ and $\varphi,\psi \in Fm^\Lambda$.

REMARK 1.1. (a) $\underline{T}$ and $\underline{F}$ denote the "TRUE" and "FALSE" formulas. We shall use $\forall v_i$, $\rightarrow$, $\leftrightarrow$ etc. as derived logical connectives.

(b) In Fm^Λ we allow only the so called "restricted" formulas, i.e. formulas in which the relational atomic formulas have a prescribed sequence of variables. (I.e.

---

*/ We assume that everything is disjoint from everything that are needed to be disjoint, e.g. $\{v_i : i\in\alpha\} \cap \{R_i : i\in\beta\} = 0$.

$R_i(v_1,v_0)$ or $R_i(v_1,v_1)\notin Fm^\Lambda$ even if $\rho_i=2$.) Allowing only the restricted formulas is not a real restriction: For $\alpha\geqslant\omega$, every first-order formula is (semantically) equivalent to a restricted one. In the present paper we mostly will have only binary relation symbols, in which case the above again holds if $\alpha\geqslant 3$. However, if $\alpha<\omega$ and there is a relation symbol of arity α then not every formula is equivalent with a restricted one. Thus allowing only restricted formulas makes our logics slightly weaker than the usual ones (e.g. those in [Poizat 82] or in [TG]). But this will make our result that "the logic with 3 variables is strong enough" even stronger (see Thm.1.6.).

(c) We do not have operation symbols in our languages. This is not a restriction from the point of view of the investigations in the present paper: One can easily express that an n+1-ary relation symbol is actually an n-ary function. See e.g. [M76]pp.205_5-208^4,Def.11.26,Thm.11.28.

(d) We required the arities to be finite numbers (i.e. that $\rho_i\in\omega$) for convenience only. The present (cylindric algebraic) approach is well suitable to investigate infinitary relations, too (i.e. where $\rho_i\in\omega$ is not required). This is illustrated e.g. in [HMT]§4.3, cf. also [Sain82], [AGN77], [N78]. $\square$

The notions of a model $\mathfrak{M}$ for Λ and that of validity $\models$, or semantical consequence $\models_\alpha$ between elements of Fm^Λ, are the usual; therefore we omit their definitions. (They can be found in [HMT]§4.3, Part II p.153.)

1.2. OUR PROOF SYSTEM

We will use the following proof system $\vdash_{r,\Lambda}$ for our languages Λ. (It coincides with a usual one for $\alpha \geq \omega$, $\vdash_{r,\Lambda}$ is defined in [HMT]§4.3, Part II p.157.)

The logical axioms $\Lambda\gamma^{\Lambda}$ are the following kind of formulas. Let $\varphi, \psi \in Fm^{\Lambda}$ and $i,j,k \in \alpha$.

((1)) φ , if φ is a propositional tautology

((2)) $\forall v_i(\varphi \rightarrow \psi) \rightarrow (\forall v_i\varphi \rightarrow \forall v_i\psi)$

((3)) $\forall v_i\varphi \rightarrow \varphi$

((4)) $\varphi \rightarrow \forall v_i\varphi$, if v_i does not occur free in φ

((5)) $v_i = v_i$

((6)) $\exists v_i(v_i = v_j)$

((7)) $v_i = v_j \rightarrow (v_i = v_k \rightarrow v_j = v_k)$

((8)) $v_i = v_j \rightarrow [\varphi \rightarrow \forall v_i(v_i = v_j \rightarrow \varphi)]$, if $i \neq j$

((9)) $\exists v_i\varphi \leftrightarrow \neg\forall v_i\neg\varphi$.

The inference rules are Modus Ponens ((MP), or detachment), and Generalization ((G)).

Let $Ax \subseteq Fm^{\Lambda}$ and $\varphi \in Fm^{\Lambda}$. We write $Ax \vdash_{r,\Lambda} \varphi$ if φ can be derived from Ax by the above proof system (in the usual sense, for more detail see p.157 of [HMT]Part II). Instead of $\vdash_{r,\Lambda}$ we shall often write $\vdash_{r,\alpha}$, $\vdash_{\alpha}$, or $\vdash_{r}$.

REMARK 1.2. Let $\Lambda = \langle \alpha, R, \rho \rangle$, β = DoR.

(a) For $\alpha \geq \omega$ we have $\vdash_{r,\alpha} = \models_{\alpha}$, i.e. our logic $\langle Fm^{\Lambda}, \models_{\alpha} \rangle$ is complete w.r.t. the proof system $\vdash_{r,\alpha}$; and also coincides with the usual first-order logic (see [HMT] 4.3.23).

(b) We call Λ monadic iff all its relation symbols are unary, i.e. iff $(\forall i\in\beta)\ \rho_i\le 1$. If Λ is monadic, then again, $\vdash_{\alpha} = \models_{\alpha}$, i.e. $\vdash_{\alpha}$ is complete w.r.t. $\models$, for any α. (For proof see Prop.1.11 in §1.5.)

(c) For $\alpha<\omega$ if Λ is not monadic then $\vdash_{\alpha}$ is not complete, i.e. $\vdash_{\alpha} \ne \models_{\alpha}$. Since clearly $\vdash_{\alpha}$ is sound, this means that there are (semantically) true formulas that are unprovable by $\vdash_{\alpha}$. (See [HMT]4.3.28+§3.2($CA_{\alpha}\ne Gs_{\alpha}$).) Below, in (E1)-(E3) we list some examples of true but unprovable formulas.

(c1) One cannot add finitely many new schemes (in the form prescribed in [M69] or in [AN80]) to the logical axioms $\Lambda_{\mathfrak{T}}^{\Lambda}$ such that $\vdash_{\alpha}$ would become complete (theorem of Monk [M69]). Thus $\models_{\alpha}$ is, in a sense, essentially incomplete (for $\alpha<\omega$). For a contrasting result see [AN81].

(c2) Let $\alpha\le\beta<\omega$. There is a valid formula using α variables which cannot be proved with β variables (theorem of Monk [M69]). That is, let $\Lambda_{\alpha} \overset{d}{=} \langle\alpha, (E), (\alpha)\rangle$ (i.e. Λ has one α-ary relation symbol E and uses α variables). Then $(\exists\,\varphi\in Fm^{\Lambda_\alpha})[\models_{\alpha}\varphi$ and <u>not</u> $\vdash_{\beta}\varphi]$. Thus for completeness, we need all the infinite variables.

Let $\varphi\in Fm^{\Lambda}$ be valid. Then there is $\beta<\omega$ such that φ is provable with β variables, by (a). We do not know whether there is a recursive function $\beta : Fm^{\Lambda} \to \omega$ such that $(\forall\,\varphi\in Fm^{\Lambda})[\models_{\alpha}\varphi \Rightarrow \vdash_{\beta(\varphi)}\varphi]$. (Problem of Biró [B85].) For partial results in this direction see [N85b].

(c3) There is a "Henkin-type, nonstandard" completeness theorem for $\vdash_{r,\alpha}$ (see Prop.1.10 in §1.5). More precisely: We can define (quite natural) nonstandard models for our languages Λ such that for every formula $\varphi \in Fm^{\Lambda}$ we have [φ can be proved by $\vdash_{\alpha}$ iff φ is valid in all, including nonstandard, models]. See Henkin [H67],[H73].

(c4) Examples of unprovable formulas. Let $\Lambda = \langle \alpha, R, \varrho \rangle$ be a language with $\alpha < \omega$.

(E1) The "merry-go-round" formulas. Let $\varphi \in Fm^{\Lambda}$. Let MGR(φ) denote the formula

$$\exists v_2(v_2=v_0 \wedge \exists v_0(v_0=v_1 \wedge \exists v_1(v_1=v_2 \wedge \exists v_2 \varphi\ ) \leftrightarrow$$
$$\exists v_2(v_2=v_1 \wedge \exists v_1(v_1=v_0 \wedge \exists v_0(v_0=v_2 \wedge \exists v_2 \varphi\ ) .$$

Intuitively, MGR(φ) expresses the equivalence of interchanging the variables v_0, v_1 in two different ways (using v_2 as auxiliary variable). Now, MGR(φ) is valid for all φ, but there are $\varphi \in Fm^{\Lambda}$ for which MGR(φ) is not $\vdash_{\alpha}$ - provable. (A result of Henkin, see [HMT]3.2.71(7).) For such a φ we can take e.g. $R(v_0 \ldots v_{\alpha-1})$ (if R is an α-ary relation symbol in Λ). We note that for every $\varphi \in Fm^{\Lambda}$, MGR(φ) can be proved with $\alpha+1$ variables, see [HMT]1.5.14.

(E2) Let $\alpha=3$. Then (I)"the composition of two functions is again a function", (II)"composition of relations is associative", (III)"the inverse of the inverse of a relation is the original relation", though expressible in Λ , are not $\vdash_{r,3}$ -provable. Again, they all are provable with 4 variables. (We note that the last sentence (III) is closely related to the merry-go-round formulas, see [HMT]Part II p.101 and [HMT] Part I p.17. Actually, (III) is equivalent with the latter.

(Results of Henkin, Maddux and Tarski, proof for (I) can be found in [Ma83], proof for (II) in [HMT]3.2.69(3), proof for (III) in [HMT]3.2.71(8).) We note that each of the above (I)-(III) express the fact that the relational-algebraic reduct of a 3-dimensional cylindric algebra is not necessarily a relation algebra in the sense of [T41],[CT51]. One of the main results of the present paper is that if we define composition and inverse of relations in a different (rather complicated but semantically correct) way then under a finite assumption, the reduct will be a relation algebra, i.e. the above (I)-(III) become $\vdash_{3}$ -provable. For more on this see Remark 2. in §2.

(E3) Let $\alpha=2$. Let R,S be binary relation symbols. Let DoR, RgR denote the domain and the range of R resp. Then the following can be expressed with a formula ψ :

"DoR=DoS; RgR=RgS; DoR is a singleton imply R=S".

A precise formulation of ψ is:

$$\forall v_0 \forall v_1 \Big[(\exists v_1 R' \leftrightarrow \exists v_1 S') \wedge (\exists v_0 R' \leftrightarrow \exists v_0 S') \wedge \\ ([\exists v_0 (v_1 = v_0 \wedge \exists v_1 R') \wedge \exists v_1 R'] \rightarrow v_0 = v_1)\Big] \rightarrow \forall v_0 \forall v_1 (R' \leftrightarrow S') ,$$

where R' is $R(v_0,v_1)$ and S' is $S(v_0,v_1)$. Then ψ is not $\vdash_{2}$ - provable. (A result of Henkin, we give a proof after Prop.1.10 in §.1.5.) Again, ψ is provable with 3 variables. □

REMARK 1.3. One might ask the question: Why are we investigating the provability relation $\vdash_{r,n}$? Is $\vdash_{r,n}$ not only one (more or less ad-hoc choice) of many possible inference systems? Would we not get completely different results if we took some other generally accepted axiomatization of first-order logic? Well, this question occurred to others (e.g. Tarski, Henkin, Maddux) in the past. Their investigation seems to indicate that the answer is no (except for some inessential minor differences such as e.g. provability of MGR , but e.g. $\not\vdash_{3}$ "associativity of composition of relations" seems to be invariant). Namely: in [M78] and [M83] Maddux investigated two inference systems both different from $\vdash_{r,n}$. The second one was not even a Hilbert style one but instead a Gentzen type sequent calculus. He found that provability with n variables remains essentially the same. Similar observations are based on Henkin [H67], see e.g. discussions of the definition of $\vdash_{n}$ on p.7 there. All these seem to justify our identifying $\vdash_{r,n}$ with provability by "the usual inference system of logic" restricted to n variables. More or less equivalent versions of $\vdash_{r,n}$ were studied e.g. in [H67],[H73],[M71], [Johnson73], [Ma78],[Ma83], [TG]. □

1.3. GÖDEL'S INCOMPLETENESS PROPERTY

The smaller α is, the weaker our first-order logic using α variables is. One measure of "strongness" of a logic is whether Gödel's incompleteness property holds for it or not.

DEFINITION 1.4. Let Λ be a language. We call a formula $\varphi \in Fm^{\Lambda}$ consistent iff not $\vdash_{r,\Lambda} \neg\varphi$. Sometimes we shall write "$\vdash_{r,\Lambda}$ -consistent" (instead of consistent) to emphasize that this is a syntactical notion. Define $Fm^{\Lambda,0} \overset{d}{=} \{\varphi \in Fm^{\Lambda} :$ φ has no free variable$\}$. Let $T \subseteq Fm^{\Lambda}$. We say that T is complete iff $(\forall \psi \in Fm^{\Lambda,0})[T \vdash_{r,\Lambda} \psi$ iff not $T \vdash_{r,\Lambda} \neg\psi]$. Define $\hat{T} \overset{d}{=} \{\varphi \in Fm^{\Lambda,0} : T \vdash_{r,\Lambda} \psi\}$. We say that <u>$\Lambda$ has Gödel's incompleteness</u> (in short, Λ has G.i.) iff there is a consistent formula $\varphi \in Fm^{\Lambda}$ that cannot be extended to a complete, decidable (i.e. recursive) theory, i.e. there is no $T \subseteq Fm^{\Lambda}$ such that $\varphi \in T = \hat{T}$, T complete, T decidable. We say that <u>Λ has weak Gödel's incompleteness</u> (Λ has w.G.i.) iff there is a consistent $\varphi \in Fm^{\Lambda}$ that cannot be extended to a finitely axiomatizable complete theory, i.e. $\neg(\exists T \subseteq Fm^{\Lambda})$ $[T$ is finite, $\varphi \in T$ and T complete$]$. $\square$

Clearly, any language with infinitely many rela-tion symbols has weak Gödel's incompleteness. However, if Λ has only finitely many relation symbols then the property of having w.G.i. is much more interesting.

We note that in all our proofs for Gödel's incompleteness, the "incompletable" formula φ will always be a consequence of the theory of <u>arithmetic</u> (in a sense).

Let $\Lambda = \langle \alpha, R, \varrho \rangle$, $\alpha = \omega$ be nonmonadic. Then Λ has G.i. by Gödel's incompleteness theorem (see e.g. [M76]Thm. 15.19, p.273) since $\vdash_{\omega}$ is a complete inference system (cf. Remark1.2(a)).

Our main theorem in the present paper is that (non-monadic) first-order logic with 3 variables has Gödel's incompleteness in the sense of Def.1.4. (i.e. it is strong enough). It was known that first-order logic with ≤ 2 variables does not have even weak Gödel's incompleteness in the case of finitely many relation symbols, not even when $\vdash_2$ is replaced with $\models_2$. (Result of Henkin, see [HMT]2.5.7(ii), 4.2.7-9.) It was asked, as Problem 4.14 in [HMT], in an algebraic form*/ whether first-order logic with $\geqslant 3$ variables has weak Gödel's incompleteness or not.

<u>REMARK 1.5.</u> In Definition 1.4 above, we defined <u>syntactic</u> notions. For $\alpha < \omega$, these syntactic notions differ from the corresponding semantical one, by Remark1.2(c). There are many $\vdash_\alpha$-consistent theories that are semantically inconsistent. E.g. by (E2) in Remark1.2(c), there are $\vdash_3$-consistent theories stating explicitly that the composition of two given functions is not a function. Thus when proving G.i. for a language Λ with $\alpha < \omega$ we have to deal with <u>semantically inconsistent theories</u>, too. Though there are more $\vdash_\alpha$ - consistent formulas than semantically consistent ones, when proving G.i. the incompletable formula φ will always be true in every model of Peano's arithmetic (in a sense), hence φ will be semantically consistent. Therefore our main theorem will imply that the semantic version of Gödel's incompleteness property holds for logic with 3 variables, too. This latter consequence is however, much easier to prove. Cf. Remark 2. , in §2. The real difficulty in proving our result (Theorem 1.6(i) below) is in dealing with those complete (hence $\vdash_{r,3}$-consistent) theories which are semantically inconsistent. $\square$

---

*/ Cf. Proposition 1.8 herein.

THEOREM 1.6. Let $\Lambda = \langle\alpha, R, \rho\rangle$ be a language, $\alpha \geqslant 3$.

(a) If Λ is not monadic, i.e. if there is at least one at least binary relation symbol in Λ, then Λ has Gödel's incompleteness.

(b) If Λ is monadic, then Λ does not have G.i. (but Λ has w.G.i. if $|DoR| \geqslant \omega$).

Proof. We prove Thm.1.6(a), as our main theorem, Thm.1, in §2. Proof of Thm.1.6(b): Assume that Λ is monadic. Let $\psi \in Fm^\Lambda$ be $\vdash_{r,\alpha}$-consistent. Then ψ has a model by Prop.1.11 in §1.5 herein. Then it is known that ψ has a finite model $\mathfrak{M}$, too. Let $T \overset{d}{=} \{\varphi \in Fm^\Lambda : \mathfrak{M} \models \varphi\}$. Now T is decidable since $\mathfrak{M}$ is finite, $\psi \in T$ by $\mathfrak{M} \models \psi$ and T is complete because T is the theory of one model and since $\vdash_{r,\alpha}$ is sound. QED

The proof we give for Thm.1.6(a) in §2 uses Tarski's QRA representation theorem (representability of relation algebras with a pair of quasi-projection elements, i.e. of QRA's). However, in §3 we outline a purely logical proof, too.

1.4. CONNECTION BETWEEN GÖDEL'S INCOMPLETENESS PROPERTY AND ATOMICITY OF THE FORMULAALGEBRAS

The ideas in this section (and their subsequent elaboration) might be related to Problem 1 (which is in §Algebraic formulation ... of ... logical results) of [M75], which asks to give an algebraic proof for Gödel's incompleteness theorem.

Clearly, G.i. $\Rightarrow$ w.G.i, but w.G.i. $\not\Rightarrow$ G.i by Thm.1.6(b). Next we characterize the property w.G.i.

DEFINITION 1.7. ([HMT]§4.3.) Let $\Lambda = \langle \alpha, R, \rho \rangle$ be a language. We define the formula-algebras ${}_p\mathfrak{Fm}^{\Lambda}$ and ${}_p\mathfrak{Fm}^{\Lambda,0}$. Let

$$\mathfrak{Fm}^{\Lambda} \overset{d}{=} \langle Fm^{\Lambda}, \vee, \wedge, \neg, \exists v_i, v_i = v_j \rangle_{i,j\in\alpha} ,$$ [*]

$$\mathfrak{Fm}^{\Lambda,0} \overset{d}{=} \langle Fm^{\Lambda,0}, \vee, \wedge, \neg \rangle$$

$${}_p\equiv^{\Lambda} \overset{d}{=} \{ \langle \varphi, \psi \rangle \in {}^2(Fm^{\Lambda}) : \vdash_{r,\alpha} \varphi \leftrightarrow \psi \} ,$$ and

$${}_p\mathfrak{Fm}^{\Lambda} \overset{d}{=} \mathfrak{Fm}^{\Lambda}/{}_p\equiv^{\Lambda} \quad , \quad {}_p\mathfrak{Fm}^{\Lambda,0} \overset{d}{=} \mathfrak{Fm}^{\Lambda,0}/{}_p\equiv^{\Lambda} \quad .$$

The subscript p intends to refer to "provability" (the algebra is formed modulo provability and not semantic equivalence). □

Clearly, ${}_p\mathfrak{Fm}^{\Lambda,0}$ is a Boolean algebra. We note that ${}_p\mathfrak{Fm}^{\Lambda,0}$ is the syntactic version (or $\vdash_{r,\alpha}$ -version) of the usual Lindenbaum-Tarski algebra of Λ.

PROPOSITION 1.8. Let Λ be a language. Then

Λ has w.G.i. $\iff$ ${}_p\mathfrak{Fm}^{\Lambda,0}$ is not atomic.

Proof. For any formula $\varphi \in Fm^{\Lambda}$, let $\bar{\varphi}$ denote its universal closure, i.e. $\bar{\varphi}$ is $\forall v_0 \dots \forall v_n \varphi$ where the free variables of φ are among $v_0, \dots, v_n$. Assume that Λ has

---

[*] Here $\vee : {}^2Fm^{\Lambda} \to Fm^{\Lambda}$ denotes the function for which $\vee(\varphi,\psi) = \varphi \vee \psi$, etc.

w.G.i. Let $\varphi \in Fm^{\Lambda}$ be a formula that cannot be extended to a finitely axiomatizable, complete and decidable theory. We will show that there is no atom below $\bar{\varphi}/{}_{p}{\equiv}^{\Lambda}$. Assume that $\tau/{}_{p}{\equiv}^{\Lambda}$ is an atom below $\bar{\varphi}/{}_{p}{\equiv}^{\Lambda}$. This means that $\tau/{\equiv} \cdot \bar{\varphi}/{\equiv} = \tau/{\equiv}$, i.e. that $\tau \wedge \bar{\varphi}/{\equiv} = \tau/{\equiv}$, i.e. $\vdash_{r,\Lambda} (\tau \wedge \bar{\varphi} \to \bar{\varphi})$, i.e. $\vdash_{r,\Lambda} \tau \to \bar{\varphi}$, where $\equiv \overset{d}{=} {}_{p}{\equiv}^{\Lambda}$. Let $T \overset{d}{=} \{\tau, \varphi\}$ and $\hat{T} \overset{d}{=} \{\psi \in Fm^{\Lambda,0} : T \vdash_{r,\Lambda} \psi\}$. We will show that T is complete and $\hat{T}$ is decidable. Let $\psi \in Fm^{\Lambda,0}$ be arbitrary. Then either $\tau/{\equiv} \leq \psi/{\equiv}$ or $\tau/{\equiv} \leq \neg\psi/{\equiv}$ since $\tau/{\equiv}$ is an atom, i.e. either $\vdash_{r,\Lambda} \tau \to \psi$ or $\vdash_{r,\Lambda} \tau \to \neg\psi$, thus either $T \vdash_{r,\Lambda} \psi$ or $T \vdash_{r,\Lambda} \neg\psi$. Both cases cannot occur since $T \vdash_{r,\Lambda} \psi$ iff $\vdash_{r,\Lambda} \tau \to \psi$ (by the deduction theorem, cf. §2, p.   and since $\vdash_{r,\Lambda} \tau \to \varphi$). Thence if $\vdash_{r,\Lambda} \tau \to \psi$ and $\vdash_{r,\Lambda} \tau \to \neg\psi$ then $\vdash_{r,\Lambda} \tau \to \underline{F}$, i.e. $\tau/{\equiv} = 0$ contradicting the fact that $\tau/{\equiv}$ is an atom. Clearly, $\hat{T}$ is recursively enumerable (since T is such). By completeness of T we have $Fm^{\Lambda,0} \sim \hat{T} = \{\neg\psi : \psi \in \hat{T}\}$, hence the complement of $\hat{T}$ is recursively enumerable, too, hence $\hat{T}$ is decidable. (We used the trivial fact that $Fm^{\Lambda,0}$ is decidable.) The proof of the converse is completely analogous: Assume that ${}_{p}\mathfrak{Fm}^{\Lambda,0}$ is not atomic. Then there is $\varphi \in Fm^{\Lambda,0}$ such that there is no atom below $\varphi/{\equiv}$. We will show that there is no finitely axiomatizable, complete and decidable extension of φ. Assume the contrary: let $\varphi \in T \subseteq Fm^{\Lambda}$, T finite, complete. Let τ be the universal closure of the conjunction $\wedge T$ of T, i.e. let $\tau = \overline{\psi_0 \wedge \ldots \wedge \psi_n}$ where $T = \{\psi_0, \ldots, \psi_n\}$. We will show that $\tau/{\equiv}$ is an atom below $\varphi/{\equiv}$. Clearly, $\tau/{\equiv} \leq \varphi/{\equiv}$ by $\varphi \in T$. Let $\psi \in Fm^{\Lambda,0}$ be arbitrary. Then either $T \vdash_{r,\Lambda} \psi$ or $T \vdash_{r,\Lambda} \neg\psi$, since T is complete, therefore either $\vdash_{r,\Lambda} \tau \to \psi$ or $\vdash_{r,\Lambda} \tau \to \neg\psi$, i.e. either $\tau/_{\equiv} \leq \psi/_{\equiv}$ or $\tau/_{\equiv} \leq \neg\psi/_{\equiv}$. $\tau/_{\equiv} \neq 0$ since e.g. $\vdash_{r,\Lambda} \tau \to \underline{T}$ iff $\nvdash_{r,\Lambda} \tau \to \underline{F}$ by ($T \vdash_{r,\Lambda} \underline{T}$ iff $T \nvdash_{r,\Lambda} \underline{F}$), hence $\nvdash_{r,\Lambda} \tau \to \underline{F}$ by $\vdash_{r,\Lambda} \tau \to \underline{T}$. QED

REMARK 1.9. We note that ${}_p\mathfrak{Fm}^{\Lambda}$ atomic $\Rightarrow$ ${}_p\mathfrak{Fm}^{\Lambda,0}$ atomic. To see this let $\varphi/_{p}{\equiv}$ be an atom in ${}_p\mathfrak{Fm}^{\Lambda}$. Since all ranks in Λ are finite, the universal closure $\bar{\varphi}/_{p}{\equiv}$ of φ is an atom in ${}_p\mathfrak{Fm}^{\Lambda,0}$. Now if b is a nonzero element in the second (the Lindenbaum-Tarski) algebra then it is such in the first one which was assumed to be atomic. Then there is an atom $\varphi/_{p}{\equiv}$ below b in the first one. But then $\bar{\varphi}/_{p}{\equiv}$ is an atom below b in the second algebra.

The other direction is not so obvious. Actually, it becomes false if we allow quotient algebras modulo arbitrary theories. (However, without theories its truth follows from our main result.) □

1.5. CONNECTIONS WITH CYLINDRIC ALGEBRAS

The title of the present paper suggests that there is a connection between the logic L_n and between the class CA_n of cylindric algebras. Indeed, there is one: CA_n's can be considered as "nonstandard models" for L_n, this way making the provability relation $\vdash_n$ complete. These "nonstandard models" can be used therefore to show unprovability of (unprovable) formulas of L_n. On the other hand, by giving criteria for a "nonstandard model" to be "standard", one can arrive at completeness results w.r.t. the original models. (This latter activity is called representation theory within CA theory). In this section we give examples of both applications of nonstandard models (i.e. of CA_n's).

The ideas in this section (application of CA's to the study of the logic L_n of n variables) are elaborated in [H73] and [H67]§4.4, pp.42-46, where Henkin starts out with L_n and arrives at CA's as the adequate tool for its study. What we call "nonstandard models" here are called generalized models therein.

Cylindric algebras are Boolean algebras enriched with some constants and unary operations such that these new constants and unary operations satisfy some additional equations. As a generic example, see ${}_p\mathfrak{Fm}^\Lambda$ in Def.1.7. (beginning of §1.4). Let α be an ordinal. Then the constants of an α-dimensional cylindric algebra (a CA_α) are denoted by d_{ij} $(i,j\in\alpha)$ and the unary functions by c_i $(i\in\alpha)$. Thus a CA_α $\mathfrak{A}$ is an algebra of the form

$$\mathfrak{A} = \langle A , +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, -^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}, c_i^{\mathfrak{A}} , d_{ij}^{\mathfrak{A}} \rangle_{i,j\in\alpha} \quad .$$

Let $\Lambda = \langle\alpha,R,\rho\rangle$ be a language with $\beta = DoR$. Let $\mathfrak{F} \stackrel{d}{=} \mathfrak{Fm}^\Lambda$. Then $\mathfrak{F}$ is an algebra similar to CA_α's where $d_{ij}^{\mathfrak{F}} = (v_i=v_j)$, $c_i^{\mathfrak{F}}(\varphi) = \exists v_i\varphi$ for any $\varphi\in Fm^\Lambda$, and $\varphi+^{\mathfrak{F}}\psi = \varphi\vee\psi$ etc. Let $\mathfrak{A}\in CA_\alpha$ and $x\in A$. Then $\Delta^{\mathfrak{A}}(x) = \{i\in\alpha : c_i^{\mathfrak{A}}x\neq x\}$. ($\Delta^{\mathfrak{A}}(x)$ simulates the set of free variables of "x".) $Zd\,\mathfrak{A} \stackrel{d}{=} \{x\in A : \Delta^{\mathfrak{A}}x=0\}$.

From our point of view, CA_α's are designed to form "nonstandard models" for the proof system $\vdash_{r,\alpha}$ as follows.

(For a detailed exposition of "this point of view" see [H73].)

Recall that $\beta = DoR$ and $\Lambda = \langle \alpha, R, \varrho \rangle$. Define $\mathcal{M}^{\Lambda} \overset{d}{=} \{ \langle \mathfrak{A}, g \rangle : \mathfrak{A} \in CA_{\alpha},\ g \in {}^{\beta}A$ and $(\forall i \in \beta)\ \Delta^{\mathfrak{A}}(g_i) \subseteq \varrho_i \}$. Let $\langle \mathfrak{A}, g \rangle \in \mathcal{M}^{\Lambda}$. Then there is a homomorphism $h : \mathfrak{Fm}^{\Lambda} \to \mathfrak{A}$ such that $(\forall i \in \beta) h(R_i(v_0 \ldots v_{\varrho_i - 1})) = g_i$ (and of course, $h(v_i = v_j) = d_{ij}^{\mathfrak{A}}$, $h(\exists v_i \varphi) = c_i^{\mathfrak{A}} h(\varphi)$, $h(\varphi \vee \psi) = h\varphi +^{\mathfrak{A}} h\psi$ etc. for every $i, j \in \alpha$ and $\varphi, \psi \in Fm^{\Lambda}$). Let $\varphi \in Fm^{\Lambda}$. We say that φ is valid in the model $\langle \mathfrak{A}, g \rangle$, in symbols $\langle \mathfrak{A}, g \rangle \models_{\alpha}^{CA} \varphi$, iff $h\varphi = 1^{\mathfrak{A}}$. ($\langle \mathfrak{A}, g \rangle$ can be thought of as an abstract model with g_i $(i \in \beta)$ as abstract relations and $+^{\mathfrak{A}}, \ldots, c_i^{\mathfrak{A}}$ as abstract disjunction,..., quantification.) We define $\models_{\alpha}^{CA} \varphi$ iff $(\forall \mathfrak{N} \in \mathcal{M}^{\Lambda})\ \mathfrak{N} \models_{\alpha}^{CA} \varphi$. We define $Ax \models_{\alpha}^{CA} \varphi$ the usual way.

Now, the equations defining CA_{α} are such that (*) below holds.

(*) Let $\mathfrak{N} \in \mathcal{M}^{\Lambda}$ be arbitrary. Then (a)-(b) below hold.

(a) $(\forall \varphi \in \Lambda_{\gamma}^{\Lambda})\ \mathfrak{N} \models_{\alpha}^{CA} \varphi$

(b) $(\forall \varphi, \psi \in Fm^{\Lambda})(\forall i \in \alpha)\Big[(\mathfrak{N} \models_{\alpha}^{CA} \varphi\ \&\ \mathfrak{N} \models_{\alpha}^{CA} \varphi \to \psi \Rightarrow \mathfrak{N} \models_{\alpha}^{CA} \psi)$ and $(\mathfrak{N} \models_{\alpha}^{CA} \varphi \Rightarrow \mathfrak{N} \models^{CA} \forall v_i \varphi)\Big]$.

By (*) we have that $\vdash_{r,\alpha} \varphi \Rightarrow \models_{\alpha}^{CA} \varphi$, for any $\varphi \in Fm^{\Lambda}$. The other direction also holds, because the equations defining CA_{α} do not say more than the above (*).

Keeping the above in mind, one can now easily obtain a set of equations defining CA_{α} (by "translating" the definition of $\vdash_{r,\alpha}$ in §.1.2 into equational form). E.g. the following set of equations will do: Let $c_i^{\partial} x \overset{d}{=} -c_i{-}x$ and $x \to y \overset{d}{=} -x + y$. $x \leq y$ stands for $x \cdot y = x$, as usual in BA theory.

C(1) equations defining BA

C(2) $c^{\partial}_i(x \to y) \leqslant (c^{\partial}_i x \to c^{\partial}_i y)$

C(3) $c^{\partial}_i x \leqslant x$

C(4) $c_i(c_i x + c_i y) = c_i x + c_i y$

$c_i(-c_i x) = -c_i x$

$c_i c_j c_i x = c_j c_i x$

$c_i d_{jk} = d_{jk}$ if $i \notin \{j,k\}$

C(5) $d_{ii} = 1$

C(6) $c_i d_{ij} = 1$

C(7) $d_{ij} \cdot d_{ik} \leqslant d_{jk}$

C(8) $d_{ij} \cdot x \leqslant c^{\partial}_i(d_{ij} \to x)$ if $i \neq j$

Now, the above C(1)-C(8) define the class CA_α.

PROPOSITION 1.10. (completeness theorem for $\vdash_{r,\Lambda}$). Let $\Lambda = \langle \alpha, R, \rho \rangle$, $Ax \subseteq Fm^\Lambda$ and $\varphi \in Fm^\Lambda$. Then

$$Ax \vdash_{r,\alpha} \varphi \quad \text{iff} \quad Ax \models^{CA}_{\alpha} \varphi . \qquad \square$$

Proposition 1.10 above together with Prop.1.8 show the connection between Gödel's incompleteness property for L_n and non-atomicity of $\mathfrak{Zd}\,\mathfrak{Fr}_\beta CA_n$, see also Remark 1.9. It also gives a tool to handle semantically inconsistent but $\vdash_n$ -consistent theories, cf. Remark 1.5.

Next, as an application of Prop.1.10, we shall show that the formula in Remark 1.2(c)(E3) is not provable by $\vdash_2$. Let $\Lambda = \langle 2, (R,S), (2,2) \rangle$ and let ψ be the formula defined in (E3). For showing $\not\vdash_2 \psi$ we will construct $\mathfrak{M} \in \mathcal{M}^\Lambda$ for which $\mathfrak{M} \not\models^{CA}_{2} \varphi$.

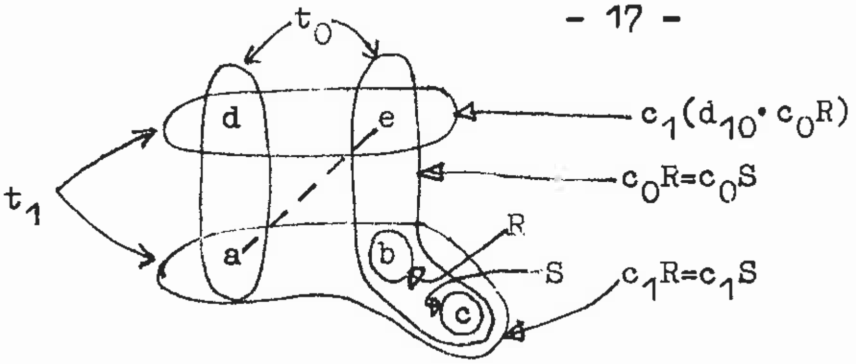


FIGURE 1

Let $U \overset{d}{=} \{a,b,c,d,e\}$, $A \overset{d}{=} \{X : X \subseteq U\}$, $+^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, -^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}$ are the Boolean set operations on A. $d_{00}^{\mathfrak{A}} \overset{d}{=} d_{11}^{\mathfrak{A}} \overset{d}{=} U$ and $d_{01}^{\mathfrak{A}} \overset{d}{=} d_{10}^{\mathfrak{A}} \overset{d}{=} \{a,e\}$. Let t_0 be the equivalence relation on U with blocks $\{d,a\}$ and $\{e,b,c\}$, and t_1 be the equivalence relation on U with blocks $\{d,e\}, \{a,b,c\}$. Let $X \in A$ and $i \in 2$. $c_i^{\mathfrak{A}} X \overset{d}{=} \{u \in U : (\exists v \in X)(u,v) \in t_i\}$.

Now it can be checked that $\mathfrak{A} \in CA_2$. (E.g. by checking the above C(1)-C(8)*/). Let $R \overset{d}{=} \{b\}$ and $S \overset{d}{=} \{c\}$ and $\mathfrak{M} \overset{d}{=} \langle \mathfrak{A}, (R,S) \rangle$. Now it is not difficult to check that

$\mathfrak{M} \underset{2}{\overset{CA}{\not\models}} \psi$. □

In any theory of "nonstandard models" it is customary to pay special attention to those "nonstandard models" which happen to be "standard". In the case of our $\mathcal{M}^{\Lambda}$ these standard objects are called representable. The reason for this is that cylindric algebras that can be obtained from real models are called <u>representable</u>. Let $\mathfrak{M}$ be a model for $\Lambda = \langle \alpha, R, \rho \rangle$. Then there is a CA_α $\mathfrak{Cs}^{\mathfrak{M}}$ naturally corresponding

---

*/ The "official" defining equations for CA_α are fewer and more easy to check. We defined $\mathfrak{A}$ by defining its so called "atom structure", i.e. by defining t_0, t_1 on U. It is even easier to check that t_0 and t_1 satisfy the few requirements for forming a cylindric atom structure, cf. [HMT] 2.7.40.

to $\mathfrak{M}$. We can define $\mathfrak{Cs}^{\mathfrak{M}}$ as follows, see [HMT]4.3.4. For any $\varphi \in Fm^{\Lambda}$ let $\tilde{\varphi}^{\mathfrak{M}} \overset{d}{=} \{ s \in {}^{\alpha}M : \mathfrak{M} \models \varphi[s] \}$. Then the universe $Cs^{\mathfrak{M}}$ of $\mathfrak{Cs}^{\mathfrak{M}}$ is $Cs^{\mathfrak{M}} \overset{d}{=} \{ \tilde{\varphi}^{\mathfrak{M}} : \varphi \in Fm^{\Lambda} \}$ and the operations are the natural ones, e.g. $\tilde{\varphi}^{\mathfrak{M}} + \tilde{\psi}^{\mathfrak{M}} = \widetilde{\varphi \vee \psi}^{\mathfrak{M}}$ $c_i^{\mathfrak{Cs}^{\mathfrak{M}}}(\tilde{\varphi}^{\mathfrak{M}}) = \widetilde{(\exists v_i \varphi)}^{\mathfrak{M}}$ etc. Now a CA_{α} is called representable iff it is a subdirect product of CA_{α}'s corresponding to models the above way. (This is the same as saying that corresponding to a set of models.) More precisely, this is the definition for $\alpha < \omega$. For $\alpha \geq \omega$ the definition is more general but for our purposes it is to this same effect[*/]. Therefore the representation theory of CA's can be used to show e.g. completeness w.r.t. the "real" semantical consequence relation $\models_{\alpha}$. As an example of this, we prove the following.

<u>PROPOSITION 1.11.</u> Let $\Lambda = \langle \alpha, R, \rho \rangle$ be monadic. Then the proof system $\vdash_{r,\alpha}$ is complete w.r.t. the semantic $\models_{\alpha}$, i.e. $\vdash_{r,\alpha} \varphi \iff \models_{\alpha} \varphi$ for any $\varphi \in Fm^{\Lambda}$.

<u>Proof.</u> Assume $\not\vdash_{r,\alpha} \varphi$. Then by Prop.1.10 there is a "nonstandard" model $\langle \mathfrak{A}, g \rangle \in \mathcal{M}^{\Lambda}$ such that $\langle \mathfrak{A}, g \rangle \not\models_{\alpha}^{CA} \varphi$. We may assume that $\{ g_i : i \in \beta \}$ generates $\mathfrak{A}$. Since Λ is monadic, each g_i is 1-dimensional in $\mathfrak{A}$, i.e. $(\forall i \in \beta)$ $\Delta^{\mathfrak{A}}(g_i) \subseteq 1$. There is a theorem of CA theory (Monk[M62]) saying that every CA_{α} generated by 1-dimensional elements is representable. Therefore $\mathfrak{A}$ is representable. This means that

---

[*/] There is a wide variety of different notions of "representability", their interconnections is investigated e.g. in [HMTAN] or in [HMT]§3.1.

there is a "real" model $\mathfrak{M}$ of Λ such that $\langle \mathfrak{A},g\rangle \not\models^{CA}_{\alpha} \varphi$ implies $\mathfrak{M} \not\models \varphi$. Thus $\not\models_{\alpha} \varphi$. The other direction follows from the soundness of $\vdash_{r,\alpha}$. <u>QED</u>

That the proof system $\vdash_{r,\alpha}$ is complete (for ordinary languages) in the case $\alpha \geqslant \omega$ can be proved the same way, using the representation theorem (of Tarski) saying that if $\alpha \geqslant \omega$ then every CA_{α} generated by finite-dimensional elements is representable. Similarly, the fact that $\vdash_{r,\alpha}$ cannot be made complete w.r.t. $\models_{\alpha}$ for $\alpha < \omega$ by adding finitely many new schemes to the logical axioms $\Lambda_{\gamma}^{\Lambda}$ (cf. Remark 1.2(c)) follows from the "nonfinitazibility" theorem of Monk [M69] saying that the representable CA_{α}'s are not finitely axiomatizable.

Representation theorems will play an important role in our proof of Gödel's incompleteness for $\alpha \geqslant 3$.

§2. THE MAIN THEOREM AND ITS PROOF

Let α, β be ordinals. Then CA_α denotes the class of all α-dimensional cylindric algebras. If $\mathfrak{A} \in CA_\alpha$ then $\mathfrak{Zd}\,\mathfrak{A}$ denotes its zero-dimensional part, i.e. $Zd\,\mathfrak{A} \overset{d}{=} \{a \in A : \Delta^{\mathfrak{A}}(a)=0\}$. Let K be a class of algebras. Then $\mathfrak{Fr}_\beta K$ denotes the β-generated free HSPK-algebra, where HSPK is the variety generated by K. Recall that Gödel's incompleteness property was defined in §1. The main result of the present paper is

THEOREM 1 (a) The logic with three variables has Gödel's incompleteness (for more precise statement see Thm.1.6 at the end of §1.3.).

(b) $\mathfrak{Fr}_1 CA_3$ is not atomic. Moreover, $\mathfrak{Zd}\,\mathfrak{Fr}_1 CA_3$ is not atomic, either.

In the course of proving Thm.1, we shall also prove the following. Recall from [HMT]Part II p.55 that $\mathfrak{Rf}(\kappa,\alpha)$ denotes the largest regular locally finite α-dimensional cylindric set algebra with base κ. (If $\alpha < \omega$ then the universe of $\mathfrak{Rf}(\kappa,\alpha)$ consists of all α-ary relations on κ.) $\mathfrak{R}(U)$ denotes the full relational set algebra with base U. IGs_α denotes the class of all representable CA_α's (if $\alpha \geqslant 2$). RA and RRA denote the classes of all relation algebras (RA's) and all representable RA's respectively. We recall from Maddux[Ma78],[Ma82] that $SA \supset RA$ is the variety of semi-associative RA's. Recall that we obtain the definition of SA by replacing associativity of the operation ";" in the definition of RA's with the weaker equation $(x;1);1=x;1$. SA's are much closer to CA_3's than RA's, cf. e.g. [Ma78]Thm.(19) p.150. The classes WA and NA were also defined in [Ma78], [Ma82] by further weakening associativity of ";" to $((x\cdot 1');1);1=(x\cdot 1');1$ and by omitting it respectively. $\overline{Eq}K$ denotes the set of equations valid in the class K of similar algebras.

<u>THEOREM 2</u> Let $\beta \geqslant 1$. (a) Let $3 \leq \alpha < \omega$. Let $K \subseteq CA_\alpha$ be such that $\overline{Eq}K$ is recursively enumerable (r.e.) and $\mathfrak{Rf}(\kappa,\alpha) \in K$ for some infinite κ. Let $\Delta : \beta \to (\alpha+1)$ be such that $Rg\Delta \nsubseteq 2$. Then $\mathfrak{Zd}\ \mathfrak{Fr}_\beta^{(\Delta)}K$ is not atomic. Hence $\mathfrak{Fr}_\beta^{(\Delta)}K$ and $\mathfrak{Fr}_\beta K$ are not atomic either. In particular, neither $\mathfrak{Fr}_\beta CA_\alpha$ nor $\mathfrak{Fr}_\beta Gs_\alpha$ is atomic.

(b) Let $K \subseteq SA$ be such that $\overline{Eq}K$ is r.e. and $\mathfrak{R}(U) \in K$ for some infinite set U. Then $\mathfrak{Fr}_\beta K$ is not atomic. In particular, neither one of $\mathfrak{Fr}_\beta SA$, $\mathfrak{Fr}_\beta RA$, $\mathfrak{Fr}_\beta RRA$ is atomic.

(c) $\mathfrak{Fr}_\beta WA$ and $\mathfrak{Fr}_\beta NA$ are atomic if $\beta < \omega$. Further, $\overline{Eq}WA$ and $\overline{Eq}NA$ are decidable.

We note that the assumption $\alpha < \omega$ in Thm.2(a) can be replaced with $Rg\Delta \subseteq \omega$ but cannot be completely omitted since $\mathfrak{Zd}\ \mathfrak{Fr}_\beta CA_\alpha$ is atomic for $\alpha \geqslant \omega$. (For proof see [N84a].) As a contrast, $\mathfrak{Fr}_\beta CA_\alpha$ is not atomic for all $\alpha \geqslant 3$ and $\beta > 0$ (for the case $\alpha \geqslant \omega$ cf. [N84a].).

<u>REMARK 2.1.</u> The proof of Thm.2(a) is not hard to generalize

to prove the following stronger result. Let $3\leq\alpha<\omega$. Let $K\subseteq Bo_\alpha$ with $\mathfrak{Rf}(\omega,\alpha)\in K$ and $\overline{Eq}K$ r.e. Assume $K\models(C_2-C_4),(C_7)$ of [HMT]p.162. Then $\mathfrak{Fr}_\beta K$ is not atomic (if $\beta>0$). Note that $K\nsubseteq CA_\alpha$ may occur in this case since $CA_\alpha\models C_1,C_5,C_6$ (but not necessarily in K). Proofidea: By $\alpha<\omega$, we have that $(C_1),(C_5),(C_6)$ is a finite set of equations containing no variables. Let $\varphi\mu(e)$ be the formula associated to the cylindric equation e as defined in §4.3 of [HMT]. Now $\varphi\mu(C_1\wedge C_5\wedge C_6)$ is a single formula and not a formula scheme. Therefore we can add this formula to Ax defined in the proof of Thm.1, obtaining say Ax^+. Then we repeat the proof of Thm.1 with Ax replaced by Ax^+. ☐

To prove Thm.s 1-2 we shall need some lemmas. Lemmas 2.2,3,6,7 are more or less known, we state and prove them for completeness and also because we shall need them in a form slightly different from the known versions. The heart of the proof is Prop. 2.10. In the proof we shall use the connection between CA's, RA's and first-order languages. We shall use the notation of [HMT], mostly the notation of [HMT]§4.3, but we shall introduce that notation wherever we need it.

In our languages mostly we shall have only binary relation symbols. In §2 we shall have only one binary relation symbol E, for convenience only. Everything in §2 can be repeated to languages having arbitrarily many binary relation symbols.

Let $2\leq\alpha\leq\omega$. Then Λ_α denotes the language (with equality) having one binary relation symbol E and having

$\{v_i : i\in\alpha\}$ as set of variables. I.e., $\Lambda_\alpha = \langle\alpha,(E),(2)\rangle$ in the notation of §1.1. Fm_α denotes the set of formulas of Λ_α, i.e. $Fm_\alpha \overset{d}{=} Fm^{\Lambda_\alpha}$ in the notation of §1.1 (this is denoted by $\Phi\mu_r^{\Lambda_\alpha}$ in [HMT]§4.3).

In what follows we shall write x,y,z instead of v_0, v_1, v_2 respectively. Throughout the paper, we shall use the following convention:

Assume that $\varphi(x,y)$ is a restricted formula with free variables among x,y and that $\varphi(x,y)$ is not in the language of the equality, i.e. that $\varphi(x,y)$ contains $E(x,y)$ as a subformula. Then

$$\text{(S)}\quad \begin{aligned} \varphi(x,z) &\overset{d}{=} \exists y(y=z \land \varphi(x,y)), & \varphi(y,z) &\overset{d}{=} \exists x(x=y \land \varphi(x,z)),\\ \varphi(y,x) &\overset{d}{=} \exists z(z=x \land \varphi(y,z)), & \varphi(z,x) &\overset{d}{=} \exists y(y=z \land \varphi(y,x)),\\ \varphi(z,y) &\overset{d}{=} \exists x(x=z \land \varphi(x,y)), & \varphi(x,x) &\overset{d}{=} \exists y(y=x \land \varphi(x,y)),\\ \varphi(y,y) &\overset{d}{=} \exists x(x=y \land \varphi(x,y)), & \varphi(z,z) &\overset{d}{=} \exists x(x=z \land \varphi(x,x)). \end{aligned}$$

We call (S) the "substitution convention"[*].

About the usage of (S): We shall have a formula abbreviated as $x_i = y_j$. Let us apply the above convention (S) to this formula ($E(x,y)$ will occur in this formula). That is, $\varphi(x,y)$ is now $x_i = y_j$. Then we shall write $\varphi(x,z)$ as $x_i = z_j$. The meaning of $x_i = z_j$ is $\exists y(y=z \land x_i = y_j)$ instead of taking the definition of $x_i = y_j$ and replacing in it y with

---

[*] We have to fix the order of substitution, because the "merry-go-round" equations are not true in CA_α, and this means that, w.r.t. provability, the order of substitution does matter. (Cf. (E1) in Remark 1.2(c), §1.2.) However, since we shall state axioms whenever we shall need them, the only important thing is to fix the order of substitution and it will not be important to know exactly how they are fixed.

z everywhere. If $\varphi(x,y)$ is $x=y$ then by $x=z$ we really mean $x=z$ and not $\exists y(y=z \wedge x=y)$ because of the requirement that $E(x,y)$ should occur in the formula $\varphi(x,y)$. Using (S) makes our formulas shorter and easier to read.

Let $H \subseteq \alpha$. Then $Fm_\alpha^H \overset{d}{=} \{\varphi \in Fm_\alpha$: all the free variables of φ are among $\{v_i : i \in H\}\}$. We shall heavily use the fact that every ordinal is the set of smaller ordinals, e.g. in the above notation H will often be an ordinal like in Fm_3^2.

Let $p_0(x,y),\ p_1(x,y) \in Fm_3^2$ be arbitrary. Given $p_i(x,y)$ $(i \in 2)$ we define $\pi \in Fm_3^0$ as follows:

$$\begin{aligned}\pi \overset{d}{=} \forall x \forall y \forall z \big[& (p_0(x,y) \wedge p_0(x,z)) \rightarrow y=z \quad \wedge \\ & (p_1(x,y) \wedge p_1(x,z)) \rightarrow y=z \quad \wedge \\ & \exists z(p_0(z,x) \wedge p_1(z,y)) \big] .\end{aligned}$$

fu. [illegible]

$z=\langle x,y\rangle$

$p_0(z,x)$ [illegible] z [illegible] pair, [illegible] x

We call π the <u>pairing formula</u>. Writing out the definition of π without using (S) would be

$$\begin{aligned}\pi = \forall x \forall y \forall z \big[& (p_0(x,y) \wedge \exists y(y=z \wedge p_0(x,y)) \rightarrow y=z \quad \wedge \\ & (p_1(x,y) \wedge \exists y(y=z \wedge p_1(x,y)) \rightarrow y=z \quad \wedge \\ & \exists z(\exists y(y=z \wedge \exists z(z=x \wedge \exists x(x=y \wedge \exists y(y=z \wedge p_0(x,y))))) \\ & \quad \wedge \exists x(x=z \wedge p_1(x,y)) \big] .\end{aligned}$$

In what follows $\models$ denotes the semantical consequence relation. The following Lemma 2.2 has been known since it states a basic property of Tarski's pairing functions. Cf. [TG], [T53].

<u>LEMMA 2.2.</u> There is a recursive function $f : Fm_\omega^2 \to Fm_3$ such that (i)-(iii) below hold for every $\varphi \in Fm_\omega^2$:

(i) $\pi \models \varphi \leftrightarrow f\varphi$

(ii) $f(\neg\varphi) = \neg f(\varphi)$

(iii) $f\varphi \in Fm_3^j$ if $\varphi \in Fm_\omega^j$, for every $j \leq 2$.

<u>Proof.</u> Let Fm_3' denote the language Fm_3 enriched with two unary (partial) function symbols p_0, p_1 and such that we consider not only restricted formulas. I.e. Fm_3' consists of all first-order formulas built up from one binary relation symbol E, two unary function symbols p_0, p_1 and using only x,y,z as variables. Let

$$\pi_p \overset{d}{=} \pi \wedge \bigwedge\{p_i(x,y) \leftrightarrow p_i(x)=y \ : \ i \in 2\}.$$

In what follows we use the validity relation $\models_p$ to denote that we use the logic where p_0, p_1 denote only <u>partial</u> functions. (For details see e.g. [Bu85]. $p_i(x)=y$ means that "p_i is defined on x and $p_i(x)=y$".) Then e.g.

$$\pi_p \models_p \forall xy\exists z(p_0z=x \wedge p_1z=y).$$

First we show the existence of $f' : Fm_\omega^2 \to Fm_3'$ with the required properties (but using "$\pi_p \models_p$" instead of "$\pi \models$").

There is an algorithm of obtaining a prenex normal form $pr(\psi)$ of $\psi \in Fm_\omega^2$ such that $pr(\psi)$ is a formula of the form $Q\,\varphi(x,y)$ where Q is a sequence of existential quantifiers and negation symbols $\neg$, $\varphi(x,y)$ is a quantifier-free formula containing only variables occurring in Q and x,y, each variable occurs only once in Q, x,y,z do not occur in Q and further $pr(\neg\psi)=\neg pr(\psi)$ for every $\psi\in Fm^2_\omega$. Let $\psi\in Fm^2_\omega$ and $pr(\psi)$ be $Q\,\varphi(x,y)$ with the above properties. Let w be a variable. Then $\varphi(x,p_0y,w/p_1y)$ denotes the formula we obtain from $\varphi(x,y)$ by replacing y,w with p_0y,p_1y respectively everywhere in $\varphi(x,y)$, simultaneously.

Assume that Q is $\nu\exists wQ'$ for some (possibly empty) sequence ν of the negation symbol and for some variable w and Q'. Then it is not difficult to check that

$$(1)\quad \pi_p \models_p \nu\exists wQ'\varphi(x,y) \;\Leftrightarrow\; \nu\exists z(p_0z=y\wedge\exists y[y=z\wedge Q'\varphi(x,p_0y,w/p_1y)]).$$

Now, based on (1) above, one can easily define the required function $f': Fm^2_\omega\to Fm'_3$ (by an obvious recursion).

Next we want to get rid of the function symbols p_0,p_1 and of the nonrestricted formulas. Recall that we have only one relation symbol E which is binary. Let $\{\bar{x},\bar{y},\bar{z}\}=\{x,y,z\}$. Let τ,σ be finite sequences of p_0,p_1 and let $i\in 2$. Then it is not difficult to check that

$$(2)\quad \pi_p \models_p E(\tau\bar{x},\sigma\bar{y}) \Leftrightarrow \exists\bar{z}[p_0\bar{z}=\tau\bar{x}\wedge p_1\bar{z}=\sigma\bar{y}\wedge E(p_0\bar{z},p_1\bar{z})]$$

$$(3)\quad \pi_p \models_p E(p_0\bar{z},p_1\bar{z}) \leftrightarrow \exists\bar{x}\exists\bar{y}[\bar{x}=p_0\bar{z}\wedge\bar{y}=p_1\bar{z}\wedge E(\bar{x},\bar{y})]$$

$$(4)\quad \pi_p \models_p \tau\bar{x}=\sigma\bar{y} \Leftrightarrow \exists\bar{z}[p_0\bar{z}=\tau\bar{x}\wedge p_1\bar{z}=\sigma\bar{y}\wedge p_0\bar{z}=p_1\bar{z}]$$

$$(5)\quad \pi_p \models_p p_i\bar{x}=\tau\bar{y} \leftrightarrow \exists\bar{z}[\bar{z}=p_i\bar{x}\wedge\bar{z}=\tau\bar{y}]$$

$$(6)\quad \pi_p \models_p \bar{x}=p_i\tau\bar{y} \Leftrightarrow \exists\bar{z}[p_i(\bar{z},\bar{x})\wedge\bar{z}=\tau\bar{y}]\;.$$

Based on (2)-(6) and on convention (S), one can define a recursive function $g: Fm'_3\to Fm_3$ such that $(\forall\varphi\in Fm'_3)$ $[\pi_p\models_p \varphi\leftrightarrow g\varphi$ and $g(\neg\varphi)=\neg g(\varphi)]$. Noticing that $\pi\models\varphi\Leftrightarrow$

$\pi_p \models_p \varphi$ for every $\varphi \in Fm_\omega$ completes the proof. QED

Let RA, Rs denote the classes of all relation algebras and all relation set algebras respectively, cf. e.g. [HMT]§5.3, [J82,84],[Ma80,82,83]. SimRA denotes the class of all algebras similar to RA's. Thus e.g. $SA \subseteq SimRA$. Let $\mathfrak{R}$ be a set. Then $Fr_{\mathfrak{R}}SimRA$ is the set of all relation algebraic terms written up from the elements of $\mathfrak{R}$ as variable symbols. I.e. $Fr_{\mathfrak{R}}SimRA$ is the universe of the free SimRA algebra generated by $\mathfrak{R}$ in accordance with the notation of [HMT]. We shall often write $RAT_{\mathfrak{R}}$, or RAT , instead of $Fr_{\mathfrak{R}}SimRA$. Thus $RAT_{\mathfrak{R}}$ is the smallest set such that (i)-(iii) below hold:

(i) $R \in RAT_{\mathfrak{R}}$ for every $R \in \mathfrak{R}$

(ii) $1', 0, 1 \in RAT_{\mathfrak{R}}$

(iii) $\tau^{\smile}$, $\tau;\sigma$, $\neg\tau$, $\tau\cdot\sigma$, $\tau+\sigma \in RAT_{\mathfrak{R}}$ if $\tau,\sigma \in RAT_{\mathfrak{R}}$.

(Here $1'$ stands for the identity relation and $^{\smile}$, ; stand for inversion and composition of relations.)

Let $X \in \mathfrak{A} \in RA$ and $\tau \in RAT_1$. Then $\tau^{\mathfrak{A}}(X)$ denotes the element $h(\tau) \in A$ where $h : \mathfrak{Fr}_1 SimRA \to \mathfrak{A}$ is the homomorphism taking the free generator of $\mathfrak{Fr}_1 SimRA$ to X. If $\mathfrak{A} \in Rs$ then $base(\mathfrak{A})$ denotes the base of $\mathfrak{A}$, cf. [HMT]5.3.2.

Our next lemma is basically the same as Lemma 5.3.12 of [HMT]*/, see also [M61b],[Ma78],[TG]. It sais, roughly, that every element of Fm_3^2 can be "expressed" with a relation algebraic term.

---

*/ We have to reprove L.5.3.12 of [HMT] because we need recursiveness and homomorphism w.r.t. $\neg$ of the translating function.

LEMMA 2.3. There is a recursive function $r : Fm_3^2 \to RAT$ such that (i)-(iii) below hold for every $\varphi \in Fm_3^2$.

(i) $(r\varphi)^{\mathfrak{A}}(X) = \{a \in 1^{\mathfrak{A}} : \langle \text{base}\,\mathfrak{A}, X\rangle \models \varphi[a]\}$ for every $X \in \mathfrak{A} \in Rs$. (Note that $\varphi[a]$ is meaningful because φ is a formula with at most two free variables v_0, v_1 .)

(ii) $r(\neg\varphi) = -r(\varphi)$.

Proof. Let RAT denote the set of all relation algebraic terms over the single variable symbol (or generator) R . That is, $RAT = RAT_{\{R\}} = Fr_{\{R\}}SimRA$. Notation: Let $\varphi \in Fm_3$ and $i, j \in 3$. Then $s^i_j\varphi \overset{d}{=} \exists v_i(v_i = v_j \wedge \varphi)$.[*/]

We define a function $t : RAT \twoheadrightarrow Fm_3^2$ as follows (see [HMT]5.3.7):

$$t(R) \overset{d}{=} E(x,y),\quad t(1') \overset{d}{=} (x=y),\quad t(1) \overset{d}{=} \underline{T}\,,\quad t(0) \overset{d}{=} \underline{F}.$$

Let $\tau, \sigma \in RAT$. Then

$$t(\tau^{\smile}) \overset{d}{=} s^2_0 s^0_1 s^1_2 t(\tau)\quad,\quad t(\tau;\sigma) \overset{d}{=} \exists v_2(s^1_2 t(\tau) \wedge s^0_2 t(\sigma)),$$

$$t(-\tau) \overset{d}{=} \neg t(\tau),\quad t(\tau\cdot\sigma) \overset{d}{=} t(\tau)\wedge t(\sigma),\quad t(\tau+\sigma) \overset{d}{=} t(\tau)\vee t(\sigma).$$

It is not difficult to check that

$$(*)\quad \tau^{\mathfrak{A}}(X) = \{a \in 1^{\mathfrak{A}} : \langle \text{base}\,\mathfrak{A}, X\rangle \models t(\tau)[a]\}\ ,$$

for every $X \in \mathfrak{A} \in Rs$ and $\tau \in RAT$.

Let $\mathcal{R} \overset{d}{=} \{X \subseteq {}^3RAT : |X| < \omega\}$. First we define a recursive function $\rho : Fm_3 \to \mathcal{R}$ with the following

---

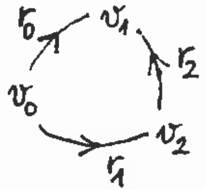
[*/] We note that if v_j does not occur in φ and $\varphi(v_i/v_j)$ denotes the formula we obtain from φ by replacing v_i everywhere by v_j then $\models s^i_j\varphi \leftrightarrow \varphi(v_i/v_j)$.

properties (i)′- (iii)′: */

(i)′ $\models \varphi \leftrightarrow \bigvee\{t(r_0)\wedge s^1_2 t(r_1)\wedge s^0_2 t(r_2) \ : \ r\in\varrho\varphi\}$, see Fig.2.

(ii)′ If $\varphi\in Fm^2_3$ then $\varrho\varphi = \{\langle\tau,1,1\rangle\}$ for some $\tau\in RAT$

(iii)′ $\varrho(\neg\varphi)=\{\langle -r_0,1,1\rangle\}$ if $\varphi\in Fm^2_3$ and $r\in\varrho\varphi$.



<u>FIGURE 2</u>

We may assume that the elements of Fm^2_3 are built up from $E(v_0,v_1)$, $v_i=v_j$ by $\vee,\neg,\exists v_i$ $(i,j\in 3)$ (since there is a recursive function transforming each element of Fm^2_3 to such a formula and preserving also the properties needed in (i)′-(iii)′).

We define ϱ by (1)-(8) below.

(1) $\varrho(E(v_0,v_1)) \overset{d}{=} \{\langle R,1,1\rangle\}$.

(2) $\varrho(v_i=v_i) \overset{d}{=} \{\langle 1,1,1\rangle\}$ for $i\in 3$

(3) $\varrho(v_0=v_1)\overset{d}{=} \varrho(v_1=v_0)\overset{d}{=}\{\langle 1',1,1\rangle\}$,
$\varrho(v_0=v_2)\overset{d}{=} \varrho(v_2=v_0)\overset{d}{=}\{\langle 1,1',1\rangle\}$,
$\varrho(v_1=v_2)\overset{d}{=} \varrho(v_2=v_1)\overset{d}{=}\{\langle 1,1,1'\rangle\}$.

Let $\varphi,\psi\in Fm_3$. Then

(4) $$\varrho(\varphi\vee\psi) \overset{d}{=} \begin{cases} \varrho\varphi\cup\varrho\psi & \text{if} \quad \varphi\vee\psi\notin Fm^2_3 \\ \{\langle \sum\{r_0 \ : \ r\in\varrho\varphi\cup\varrho\psi\},1,1\rangle\} & \text{if } \varphi,\psi\in Fm^2_3 \end{cases}$$

---

*/ Intuitively, (i)′ means that every formula $\varphi\in Fm^2_3$ can be "decomposed" into a Boolean combination of "simple" formulas $\psi(v_i v_j)$, $i,j\in 3$, where "simple" means that ψ is obtained from a relation algebraic term. This is true because we have only binary relation symbols and only 3 variables. If we have only binary relations but 4 variables, then e.g. $\exists v_3(R(v_0v_3)\wedge R(v_1v_3)\wedge R(v_2v_3))$ cannot be decomposed in the above way. This is proved in [N85a], for more on this see Remark2.5.

$$(5)\quad \varrho(\neg\varphi) \overset{d}{=} \begin{cases} \{\langle \prod\{-r_0 : r\in H_0\}, \prod\{-r_1 : r\in H_1\}, \prod\{-r_2 : r\in H_2\}\rangle \\ \quad : H_0\cup H_1\cup H_2 = \varrho\varphi,\ H_i\cap H_j = 0 \text{ for } i<j<3\} \text{ if } \varphi\notin Fm_3^2 \\ \{\langle -r_0,1,1\rangle\} \quad \text{if} \quad \varphi\in Fm_3^2 \quad \text{and} \quad \varrho\varphi=\{r\}\ . \end{cases}$$

$$(6)\quad \varrho(\exists v_2\varphi) \overset{d}{=} \{\langle \sum\{r_0\cdot(r_1;r_2) : r\in\varrho\varphi\},1,1\rangle\}$$

$$(7)\quad \varrho(\exists v_0\varphi) \overset{d}{=} \begin{cases} \{\langle 1,1,\sum\{r_2\cdot(r_1^{\smile};r_0) : r\in\varrho\varphi\}\rangle\} \text{ if } \varphi\notin Fm_3^2 \\ \{\langle 1;\sum\{r_2\cdot(r_1^{\smile};r_0) : r\in\varrho\varphi\},1,1\rangle\} \text{ if } \varphi\in Fm_3^2 \end{cases}$$

$$(8)\quad \varrho(\exists v_1\varphi) \overset{d}{=} \begin{cases} \{\langle 1, \sum\{r_1\cdot(r_0;r_2^{\smile}) : r\in\varrho\varphi\},1\rangle\} \text{ if } \varphi\notin Fm_3^2 \\ \{\langle (\sum\{r_1\cdot(r_0;r_2^{\smile}) : r\in\varrho\varphi\});1,1,1\rangle\} \text{if } \varphi\in Fm_3^2 \end{cases}$$

Now $\varrho : Fm_3 \to \mathfrak{R}$ is clearly recursive, since Fm_3^2 is a recursive subset of Fm_3, and it is not difficult to check that (i)′-(iii)′ hold. Let $\varphi\in Fm_3^2$. Then we define

$$r(\varphi) \overset{d}{=} \tau \quad \text{where} \quad \varrho\varphi = \{\langle\tau,1,1\rangle\}\ .$$

Then $r : Fm_3^2 \to RAT$ is recursive. Also, $\models \varphi \leftrightarrow t(r\varphi)$ by (i)′-(ii)′, hence (i) holds by (∗). (ii) holds by (iii)′.

<u>QED</u>

<u>REMARK 2.4.</u> In §2 we will not use the exact form of f or r . We will use only the stated properties of $f\circ r$, namely that

(1) $f\circ r : Fm_\omega^2 \to RAT$ is recursive,

(2) $f\circ r$ "preserves meaning" (see L.2.2(i)+L.2.3(i)) and

(3) $f\circ r$ preserves negation, i.e. $f\circ r(\neg\varphi) = -f\circ r(\varphi)$.

In §3 we will use two more properties of $f\circ r$, namely that

(4) $f\circ r$ preserves disjunction, and

(5) $f\circ r(E(x,y)) = E$ (for all relation symbols E).

There are methods different from ours to achieve (1)-(5). For example: Assume that there is F that satisfies Lemma 2.2(i), (iii) (i.e. F satisfies the "f"-part of (1)-(2) above). Then one can define f' which satisfies (3)-(5) in addition, too, as follows. We define f' by induction on the formulas:

$$f'(E(x,y)) \overset{d}{=} E(x,y) \ ,$$
$$f'(v_i=v_j) \overset{d}{=} F(v_i=v_j) \ ,$$
$$f'(\neg\varphi) \overset{d}{=} \neg f'(\varphi), \quad f'(\varphi \vee \psi) \overset{d}{=} f'\varphi \vee f'\psi, \quad f'(\varphi \wedge \psi) \overset{d}{=} f'\varphi \wedge f'\psi$$
$$f'(\exists v_i\varphi) \overset{d}{=} F(\exists v_i\varphi).$$

Then it can be checked that f' will satisfy (1)-(5) above. The same thing can be done for the "r-part" of (1)-(5) above. □

<u>REMARK 2.5.</u> Lemma 2.3 above does not extend from $Fm_3^2 \to RAT$ to $Fm_4^2 \to RAT$ as Thm.2.5.1 below together with the discussion preceding it shows. Namely, the algebraic form of Lemma 2.3 concernes the relationship between CA_3's and RA's. The investigation of the relationship between CA_3's and RA's goes back to well before 1961: Some years before 1961 Tarski conjectured that the study of RA's can be reduced to the study of some class of CA_3's, cf. [M61b]p.51^{7-11}. This gave rise to the problem of finding that class of CA_3's. In this direction [M61b] proved

$$(*) \quad RA = \mathfrak{Nr}^{*}CA_3''$$

for a certain class $CA_3'' \subseteq CA_3$. (The direction $RA \supseteq \mathfrak{Nr}^{*}CA_3''$ was obtained by Gebhard Fuhrken.) The definition of CA_3''

referred explicitly to the operations of $\mathfrak{Ra}^*CA_3$, therefore it was natural to try to substitute CA_3'' in the above (*) with some more natural subclass of CA_3. Henkin and Tarski proved that $Nr_3CA_\alpha \subseteq CA_3''$ for $\alpha \geqslant 4$ ([M61b]Thm.9.2,[HMT]5.3.8), therefore $\mathfrak{Ra}^*Nr_3CA_\alpha \subseteq RA$ for $\alpha \geqslant 4$ giving rise to Nr_3CA_α as a candidate for a natural substitute for CA_3''. But Monk showed that $RA \not\subseteq \mathfrak{Ra}^*Nr_3CA_\alpha$ for $\alpha \geqslant 5$ ([M61b]Thm.9.16,[M61a] Thm.2). The problem arose whether $RA \subseteq \mathfrak{Ra}^*Nr_3CA_4$ or not. This is Problem 5 in [M61b]p.80; an equivalent form of this is whether $RA = \mathfrak{Ra}^*CA_4$ or not. A partial positive solution to this 1961 problem of Monk was found by Maddux, namely [Ma78]Thm.21, which is quoted as Thm.5.3.17 in [HMT], says $RA = \mathbf{S}\mathfrak{Ra}^*CA_4$. To answer Monk's original question amounts to deciding whether or not $\mathbf{S}$ can be dropped in the preceding equation. This problem is asked in Maddux [Ma78] on p.151 (immediately above Thm.20). In [N85a] the following negative solution is proved:

<u>THEOREM 2.5.1.</u> $RA \neq \mathfrak{Ra}^*CA_4$, therefore $RA \neq \mathfrak{Ra}^*Nr_3CA_4$ either.

Thus the answer to Problem 5 in [M61b] is no. This improves Thm.9.16 of [M61b] as well as Thm.2 of [M61a]. As a corollary of Thm.2.5.1 above, we conclude that Thm.7 on p.133 of [Ma78] does not generalize from $\mathfrak{A} \in CA_3'$ to $\mathfrak{A} \in CA_3$, and [HMT] 5.3.12 does not generalize from SNr_3CA_4 to Nr_3CA_4. (These corollaries, however, have easier proofs, cf. [N85a].) □

Let $\lambda \in Fm_\omega^0$. We call λ <u>inseparable</u> iff there is no set $T \subseteq Fm_\omega^C$ which recursively separates the theorems of λ

from the refutable sentences of λ, i.e. iff there is no recursive (i.e. decidable) set $T \subseteq Fm^0_\omega$ such that $\{\varphi \in Fm^0_\omega : \lambda \models \varphi\} \subseteq T \subseteq \{\varphi \in Fm^0_\omega : \lambda \not\models \neg\varphi\}$. Cf. [M76]Def.15.7, p.266.

Let $p \overset{d}{=} r(p_0(x,y))$, $q \overset{d}{=} r(p_1(x,y))$ and[*/]

$$\pi_{RA} \overset{d}{=} (p^{\smile};p \to 1')\cdot(q^{\smile};q \to 1')\cdot(p^{\smile};q).$$

Then $\pi_{RA} \in RAT$ (since $p_i(x,y) \in Fm^2_3$).

LEMMA 2.6. Let $\lambda \in Fm^0_\omega$ be inseparable and let $\eta \overset{d}{=} (rf\lambda)\cdot\pi_{RA}$. Then there is no decidable proper congruence $R \in Co\,\mathfrak{Fr}_1SimRA$ such that $\eta \in 1/R$ and $\mathfrak{Fr}_1SimRA/R \in RA$.

Proof. Assume R is such. Define $T \overset{d}{=} \{\varphi \in Fm^0_\omega : rf\varphi \in 1/R\}$. We will show that T recirsively separates the theorems of λ from the refutable sentences of λ which contradicts the choice of λ. T is recursive because r, f and R are decidable. Assume $\gamma \in Fm^0_\omega$ is such that $\lambda \models \gamma$. We will show that $\gamma \in T$. Let $\mathfrak{F} \overset{d}{=} \mathfrak{Fr}_1SimRA/R$. Then $\mathfrak{F} \in RA$ and $\bar{p} \overset{d}{=} p/R$, $\bar{q} \overset{d}{=} q/R$ are "pairing functions" in $\mathfrak{F}$ by $\pi_{RA} \in 1/R$. Therefore $\mathfrak{F}$ is representable, i.e. $\mathfrak{F} \in RRA$ by Tarski's theorem $QRA \subseteq RRA$, see [Ma78a]. Assume $\gamma \notin T$. This means $rf\gamma \notin 1/R$. By $\mathfrak{F} \in RRA$ and $\eta \in 1/R$ then there are $\mathfrak{A} \in Rs$ and $Z \in A$ such that $\eta^{\mathfrak{A}}(Z)=1^{\mathfrak{A}}$ while $(rf\gamma)^{\mathfrak{A}}(Z) \neq 1^{\mathfrak{A}}$. By $\eta = (rf\lambda)\cdot\pi_{RA}$ we also have $(rf\lambda)^{\mathfrak{A}}(Z)=1^{\mathfrak{A}}$ and $\pi_{RA}^{\mathfrak{A}}(Z)=1^{\mathfrak{A}}$. Let $U \overset{d}{=} base\,\mathfrak{A}$ and $\mathfrak{M} \overset{d}{=} \langle U,Z\rangle$. By Lemma 2.3 we then have

---

[*/] Here a → b abbreviates $-a+b$ as usual in Boolean algebra theory.

(*) $\mathfrak{M} \models f\lambda$, $\mathfrak{M} \not\models f\gamma$ and

(**) $p^{\mathfrak{A}}(Z) = \{\langle u,v\rangle \in {}^2U : \mathfrak{M} \models p_0(u,z)\}$,

$q^{\mathfrak{A}}(Z) = \{\langle u,v\rangle \in {}^2U : \mathfrak{M} \models p_1(u,z)\}$.

By $\pi^{\mathfrak{A}}_{RA}(Z) = 1^{\mathfrak{A}}$ and (**) we then have $\mathfrak{M} \models \pi$. By Lemma 2.2 and (*) then $\mathfrak{M} \models \lambda$, $\mathfrak{M} \not\models \gamma$, contradicting $\lambda \models \gamma$. Thus $\gamma \in T$ for every $\gamma \in Fm^0_\omega$ for which $\lambda \models \gamma$. I.e. T contains the theorems of λ. Assume $\lambda \models \neg\gamma$ for some $\gamma \in Fm^0_\omega$. We will show $\gamma \notin T$. We have $\neg\gamma \in T$ by $\lambda \models \neg\gamma$, i.e. $rf(\neg\gamma) \in 1/R$. But $rf(\neg\gamma) = -rf(\gamma)$ by Lemma 2.2(ii) and Lemma 2.3(ii), hence $rf(\gamma) \in 0/R \neq 1/R$, i.e. $rf(\gamma) \notin 1/R$. This means $\gamma \notin T$. Thus T is disjoint from the refutable sentences of λ. <u>QED</u>

Let $\pi' \overset{d}{=} \pi \wedge \forall xyzv_3[(p_0(z,x) \wedge p_1(z,y) \wedge p_0(v_3,x) \wedge p_1(v_3,y)) \rightarrow z = v_3]$. $z=\langle x,y\rangle$ $v_3=\langle x,y\rangle$ $\Rightarrow z=v_3$

Then $\pi' \in Fm^0_4$. π' expresses that "the pair is unique".

[illegible] $T \subseteq Fm^0_\omega$ [illegible] $Thm(\lambda) \subseteq T \subseteq$ [illegible]$(\neg\lambda)$

$Thm(\lambda)$ $\{\varphi \in Fm^0_\omega :$ [illegible] $\lambda \not\models \neg\varphi\}$

<u>LEMMA 2.7.</u> There exist an inseparable $\lambda \in Fm^0_\omega$ and $p_i(x,y) \in Fm^2_3$ $(i \in 2)$ such that λ is semantically consistent with π', i.e. $\lambda \wedge \pi'$ has a model.

<u>Proof.</u> Recall the finite set A_E of axioms for arithmetic*/ from [E72]p.194. Then A_E is inseparable by Exercise 1 in

---

*/ We could take any variant of Robinson's arithmetic which is finitely axiomatizable and at the same time inseparable, see e.g. [M76]Thm.16.1,p.280 saying Q is inseparable and Def.14.17 (or Prop.14.18) saying that Q is finitely axiomatizable. (Warning: "essentially undecidable" is weaker than "inseparable" and is not enough for our purposes, see e.g. [M76]p.269, middle of page.) For recently found "minimal" versions of this theory with the desired properties see e.g. [Shep83].

[E72]p.238. Our inseparable formula λ will be A_E translated to set theory (and relativized to the finite ordinals), while $p_i(x,y)$ $(i\in 2)$ will be formulas in set theory expressing the usual intended meaning. Then λ will be inseparable and the model $\mathcal{H}=\langle H,\in\rangle$ of all hereditarily finite sets will be a model of $\lambda\wedge\pi'$.

The definition of $p_i(x,y)$ for $i\in 2$:

For convenience, we shall write xEy instead of $E(x,y)$.

$x=\{y\} \overset{d}{=} yEx\wedge\forall z(zEx\to z=y)$,

$\{x\}Ey \overset{d}{=} \exists z(z=\{x\}\wedge zEy)$

$x=\{\{y\}\} \overset{d}{=} \exists z(z=\{y\}\wedge x=\{z\})$

$xE\bigcup y \overset{d}{=} \exists z(xEz\wedge zEy)$

$$\mathrm{pair}(x) \overset{d}{=} \exists y\big[\{y\}Ex\wedge\forall z(\{z\}Ex\to z=y)\big]\wedge\forall zy\big[(zE\bigcup x\wedge\{z\}\not Ex\wedge yE\bigcup x\wedge\{y\}\not Ex)\to z=y\big]\wedge\forall zEx\exists y(yEz).$$

$p_0(x,y) \overset{d}{=} \mathrm{pair}(x)\wedge\{y\}Ex$

$p_1(x,y) \overset{d}{=} \mathrm{pair}(x)\wedge[x=\{\{y\}\}\vee(\{y\}\not Ex\wedge yE\bigcup x)]$.

$p_0(x,y)$, $p_1(x,y)\in Fm_3^2$ have been defined.

The formulation of λ (we shall be more sketchy here):

$xEOrd \overset{d}{=}$ "x is transitive and E is a total ordering on x"

$xEFord \overset{d}{=} xEOrd\wedge$"every element of x is a successor ordinal"

$x=0 \overset{d}{=}$ "x has no element"

$sx=z \overset{d}{=} z=x\cup\{x\}$

$x\leq y \overset{d}{=} x\subseteq y$, $x<y \overset{d}{=} x\leq y\wedge x\neq y$,

$x+y=z \overset{d}{=} \exists v(z=x\cup v\wedge x\cap v=0\wedge$"there is a bijection between v and y")

$x\cdot y=z \overset{d}{=}$ "there is a bijection between z and $x\times y$"

$x\underline{\mathrm{exp}}y=z \overset{d}{=}$ "there is a bijection between z and the set of all functions from y to x"

Then λ' is the formula saying: "0,s,+,·,$\underline{exp}$ are functions of arities 0,1,2,2,2 resp. on Ford" and $(\forall xy \in Ford)[sx\neq 0 \wedge (sx=sy \to x=y) \wedge (x<sy \leftrightarrow x\leq y) \wedge x\not< 0 \wedge (x<y \vee x=y \vee y<x) \wedge x+0=x \wedge x+sy=s(x+y) \wedge x\cdot 0=0 \wedge x\cdot sy=x\cdot y+x \wedge x\underline{exp}0=s0 \wedge x\underline{exp}sy= x\underline{exp}y\cdot x]$."

Now $\lambda\in Fm^0_\omega$ is defined to be the restricted form of the above λ' (cf. [HMT]4.3.6.). QED

REMARK 2.8. Now we have all the tools needed to prove Gödel's incompleteness property for nonmonadic languages using ≥ 4 variables. (This shows that the $\alpha>3$ case is much easier than the $\alpha=3$ case.) The idea is the following. The "relation algebraic reduct" $\mathfrak{Ra}\,\mathfrak{Fm}_\alpha$ of $\mathfrak{Fm}_\alpha$ is defined in a natural way on Fm^2_α. E.g. $\varphi;\psi \overset{d}{=} \exists z(\varphi(x,z)\wedge\psi(z,y))$, cf. [HMT]5.3.7. For $\alpha\geq 4$ $(\mathfrak{Ra}\,\mathfrak{Fm}_\alpha)/_{p\equiv} \in RA$ by [HMT]5.3.8 since ${}_p\mathfrak{Fm}_\alpha \in CA_\alpha$ by [HMT]4.3.22. Let $\mathfrak{G} \overset{d}{=} \mathfrak{Fr}_1 SimRA$ and let $h:\mathfrak{G}\to\mathfrak{Ra}\,\mathfrak{Fm}_\alpha$ be a homomorphism taking the free generator of $\mathfrak{G}$ to $E(x,y)$. Let $\psi \overset{d}{=} h\eta$, where η is $rf\lambda\cdot\pi_{RA}$ as in Lemma 2.6 and λ is inseparable such that $\lambda\wedge\pi'$ is consistent, cf. Lemma 2.7. Then ψ will be semantically consistent (since h "preserves meaning"). Assume that T is a decidable complete theory containing ψ. Define $R \overset{d}{=} \{(\tau,\sigma)\in {}^2G : (h\tau\leftrightarrow h\sigma)\in T\}$. Then clearly R is decidable since T is such, it can be seen that R is a congruence on $\mathfrak{G}$, $\eta\in 1/R\neq 0/R$ and $\mathfrak{G}/R\in RA$ by $\mathfrak{Ra}\,\mathfrak{Fm}_\alpha/_{p\equiv}\in RA$. These contradict Lemma 2.6, hence there is no such T. This proof, however, does not work for $\alpha=3$, since the relation algebraic reduct, though can be defined for $\mathfrak{Fm}_3$

(i.e. for CA_3), is not an RA. (E.g. ";" is not associative but also both $x^{\smile\smile}=x$ and the Peircean law fail in $\mathfrak{Ra}^*CA_3$. Further $x^{\smile};(-x)\leq -1'$ fails, too.) Until now, no generalized reduct of CA_3 has been known which was an RA, not even under assuming finitely many axioms in Fm_3. The essential part of our proof for $\alpha=3$ will be the definition of such a reduct. The main idea is to use Fm^1_α instead of Fm^2_α as the universe of the RA - this way we will have 2 auxiliary variables even in the case $\alpha=3$. However, we shall have to be careful to remain within Fm^1_α even when writing up the auxiliary definitions, otherwise the idea does not work. We will code binary relations as unary ones with the help of the pairing functions $p_i(x,y)$. Our main effort will go into using only finitely many axioms (in the language of Fm_3). (It is not trivial that this can be done since we have to prove "schemes" like associativity of relation composition.)

We note that now we have all the tools to prove that free relation algebras are not atomic (using the above argument, we do not even have to define RA-reduct). However, we shall prove these later, after the proof of our main result.

We also note that using Lemmas 2.2,6 one can prove that the semantic version of Gödel's incompleteness property holds for languages using ≥ 3 variables, hence that $\mathfrak{Fr}_\beta Gs_\alpha$ is not atomic for any $\beta\geq 1$ and $\alpha\geq 3$. What we said above about the $\alpha\geq 4$ case proves that $\mathfrak{Fr}_\beta CA_\alpha$ for $\alpha\geq 4$ is not atomic, by §1.4. □

We are going to define a (generalized) reduct of a relativization of the free CA_3 which is a relation algebra. This is the most important part of our proof. We shall work mostly "on the logic side" for a while.

First we define some auxiliary formulas. Let 2^* denote the set of all finite sequences of $0,1$ including the empty sequence $\langle\rangle$ as well. If $i,j\in 2^*$ then ij denotes their "concatenation" usually denoted by $i^\frown j$, and $|i|$ denotes the "length" of i. Further, if $k\in 2$ then we write k instead of $\langle k\rangle$ for the sequence $\langle k\rangle$ of length 1.

We are going to define formulas $(x_i=y_j)\in Fm_3$ for[*/] $i,j\in 2^*$. (We shall need these formulas only for $|i|,|j|\leq 3$.) Recall our convention (S) from the beginning of the proof. We write $x_i=y$ and $x=y_i$ for $x_i=y_{\langle\rangle}$ and $x_{\langle\rangle}=y_i$ resp. if $i\in 2^*$. Let $i,j\in 2^*$ and $k\in 2$. Then

$$(x_{\langle\rangle}=y) \overset{d}{=} (x=y) \quad ,$$

$$(x_k=y) \overset{d}{=} p_k(x,y) \quad ,$$

$$(x_{ik}=y) \overset{d}{=} \exists z(x_i=z \wedge z_k=y) \quad ,$$

$$(x_i=y_j) \overset{d}{=} \exists z(x_i=z \wedge y_j=z) \quad ,$$

$$(x=y_j) \overset{d}{=} (x_{\langle\rangle}=y_j) \quad .$$

Let $\varphi\in Fm_3^1$, $u\in\{y,z\}$ and $i\in 2^*$. Then $\varphi u_i \overset{d}{=} \exists x(x=u_i \wedge \varphi)$.

[*/] The intended meaning of $(x_{i_0\dots i_n}=y_{j_0\dots j_k})$ is that if p_0,p_1 are partial functions then $p_{i_n}\dots p_{i_0}x = p_{j_k}\dots p_{j_0}y$. Any restricted formula using only 3 variables and expressing this, will do.

DEFINITION 2.9. (The definition of $\mathfrak{Ora}$)

We define some new operations on Fm^1_3. Let pair(x) $\overset{d}{=}$ $\exists y p_0(x,y) \wedge \exists y p_1(x,y)$. Let $\varphi,\psi \in Fm^1_3$. Then we define

$$\varphi \odot \psi \overset{d}{=} \text{peir}(x) \wedge \exists y(\varphi y_0 \wedge \psi y_1 \wedge x_0 = y_{00} \wedge x_1 = y_{11} \wedge y_{01} = y_{10}) \quad ,$$

see Figure 3.

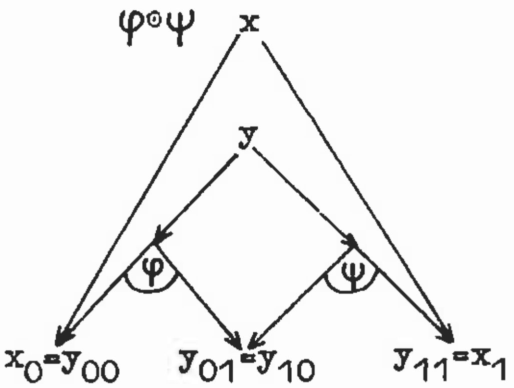


FIGURE 3

(Illustration of $\varphi \odot \psi$.)

$\varphi^{\smile} \overset{d}{=} \text{pair}(x) \wedge \exists y(\varphi y \wedge x_0 = y_1 \wedge x_1 = y_0)$,

$1' \overset{d}{=} \text{pair}(x) \wedge x_0 = x_1$,

$1 \overset{d}{=} \text{pair}(x)$, $\quad 0 \overset{d}{=} \underline{F}$

$\varphi + \psi \overset{d}{=} \text{pair}(x) \wedge (\varphi \vee \psi)$, $\qquad \varphi \cdot \psi \overset{d}{=} \text{pair}(x) \wedge \varphi \wedge \psi$,

$\dot{-}\varphi \overset{d}{=} \text{pair}(x) \wedge \neg \varphi$.

$\text{Ora} \overset{d}{=} \{\text{pair}(x) \wedge \varphi \ : \ \varphi \in Fm^1_3\}$, $\mathfrak{Ora} \overset{d}{=} \langle \text{Ora}, +, \cdot, \dot{-}, 0, 1, \odot, {}^{\smile}, 1' \rangle$.

Then $\mathfrak{Ona} \in$ SimRA. We call $\mathfrak{Ona}$ the one-dimension-based RA, where "one-dimension-based" refers to our efforts to use only one variable in defining the relation algebraic operations. $\square$

Recall from §1.2 the proof system $\vdash_{r,\alpha}$ which we denote by $\vdash_r$ as well. Define

$$p\equiv^{\alpha}_{Ax} \overset{d}{=} \equiv_{Ax} \overset{d}{=} \{(\varphi,\psi) \in {}^2Fm_\alpha : Ax \vdash_r \varphi \leftrightarrow \psi\}, \text{ and}$$

$$\mathfrak{Fm}_\alpha \overset{d}{=} \mathfrak{Fm}_r^{\Lambda\alpha} \overset{d}{=} \langle Fm_\alpha, \vee, \wedge, \neg, \underline{T}, \underline{F}, \exists v_i, v_i = v_j \rangle_{i,j\in\alpha}.$$

(Cf. §1.4 and [HMT]§4.3.) Then $\mathfrak{Fm}_\alpha$ is an algebra similar to CA_α's. The basic fact we shall use about $\vdash_r$ is that $\mathfrak{Fm}_\alpha/\equiv_{Ax} \in CA_\alpha$ for any Ax. We note that if Ax is finite then $\mathfrak{Fm}_\alpha/\equiv_{Ax} \cong \mathfrak{Rl}_a \mathfrak{Fr}_1 CA_\alpha$ for some $a \in Fr_1 CA_\alpha$.

PROPOSITION 2.10. There is a finite $Ax \subseteq Fm_3^0$ such that

$\mathfrak{M}' \models Ax$ and $\mathfrak{Ora}/\equiv_{Ax} \in RA$.

Proof. About defining Ax: We do not define Ax before the proof. Instead, during the proof we shall postulate that some formulas are elements of Ax, hence Ax will be listed during the proof. We shall be careful to keep Ax finite, and check $\mathfrak{M}' \models Ax$ but otherwise Ax will be very redundant: it could be reduced to a few, natural axioms about $p_0(x,y)$, $p_1(x,y)$.

Assume that Ax is given. By [HMT] 4.3.20 we have $\equiv_{Ax} \in Co\,\mathfrak{Fm}_x$, hence $\equiv_{Ax} \in Co\,\mathfrak{Ora}$ as well, since all the operations of $\mathfrak{Ora}$ are polynomials in $\mathfrak{Fm}_x$. Let $\mathfrak{R} \overset{d}{=} \mathfrak{Ora}/\equiv_{Ax}$. We have to show that $\mathfrak{R} \in RA$. We shall prove conditions (1)-(4) on p.162 of [Ma 78a] (which is the same as (i)-(iv) of Def.4.1 in [JT52] or Thm.2.4(i)-(iii) in [J82]). Obviously, $\mathfrak{R} \in BA$, hence condition (1) is satisfied. Next we shall prove condition (2), i.e. we shall show that $\odot$ is associative in $\mathfrak{R}$.

In order to prove associativity of $\odot$ in $\mathfrak{Ora}/\equiv_{Ax}$, we have to prove

$Ax \vdash_r (\varphi \odot \psi) \odot \tau \leftrightarrow \varphi \odot (\psi \odot \tau)$, for every $\varphi, \psi, \tau \in Ora$.

About the proof system $\vdash_r$ we shall use only the following facts.

(*) $\mathcal{F}m_\alpha/{\equiv_{Ax}} \in CA_\alpha$ (cf. [HMT] 4.3.22) and

(**) $\vdash_r \varphi \leftrightarrow \exists v_i\varphi$ if v_i does not occur freely[*/] in $\varphi\in Fm_\alpha$.

Let $u,w\in\{x,y,z\}$, $i,j\in 2^*$ and $\varphi\in Fm_3^1$. Then the following (***) is easy to check:

(***) the only free variable of φu_i is u and the free variables of $u_i=w_j$ are u,w.

Note that by (*) we have e.g.

$Ax \vdash_r \exists z\varphi \leftrightarrow \exists z(y=z \wedge \exists z\varphi)$ for every $\varphi\in Fm_\alpha$ because $CA_\alpha \models c_2X=c_2(d_{12}\cdot c_2X)$.

We shall write "by CA" when we use (*), "by FO" (by free occurrence) when we use (**),(***), and we shall write "by Ax" when we use the fact that a certain formula is in Ax. Often, we will omit the explanation "by CA".

In the proofs we shall use the fact that the refined deduction theorem holds for $\vdash_r$, i.e. that if $Ax\cup\{\varphi\} \vdash_r \psi$ without using the rule of generalization then $Ax \vdash_r (\varphi \rightarrow \psi)$. (Compare [HMT] p.$161_4$.) Using this deduction theorem makes the proofs much shorter; however, each proof will be easy to write out without using the refined deduction theorem.

[*/] To see this, we have $\vdash_r \varphi\rightarrow\exists v_i\varphi$ by (*). Assume v_i does not occur freely in φ. Then $\vdash_r \neg\varphi\rightarrow\neg\exists v_i\neg\neg\varphi$ by ⟦4⟧ and ⟦9⟧ in the definition of $\vdash_r$, hence $\vdash_r \exists v_i\varphi\rightarrow\varphi$ by ⟦1⟧ and (MP).

From now on, let $\varphi,\psi,\gamma \in \mathrm{Ora}$ be arbitrary.

Let $\{u,w\}=\{z,y\}$ (i.e. $u=z, w=y$ or $u=y, w=z$) and $i,j\in 2^{*}$, $|i|, |j| \leq 2$.

(1) $\mathrm{Ax} \vdash_{r} \varphi u_i \wedge u=w_j \rightarrow \varphi w_{ji}$ and

$\mathrm{Ax} \vdash_{r} \varphi u_i \wedge u_i=w_j \rightarrow \varphi w_j$.

For, $\varphi u_i \wedge u=w_j \vdash_{r}$ (by definition)

$\exists x(x=u_i \wedge \varphi) \wedge u=w_j \vdash_{r}$ (by FO)

$\exists x(x=u_i \wedge \varphi) \wedge \exists x(u=w_j) \vdash_{r}$ (by $\mathrm{CA} \models c_0X\cdot c_0Y=c_0(X\cdot c_0Y)$, FO)

$\exists x(x=u_i \wedge \varphi \wedge u=w_j) \vdash_{r}$ (by $\mathrm{CA} \models X\cdot Y=Y\cdot X$)

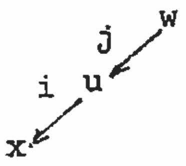
$\exists x(x=u_i \wedge u=w_j \wedge \varphi) \vdash_{r}$ (by Ax1)

$\exists x(x=w_{ji} \wedge \varphi) \vdash_{r}$ (by definition)

φw_{ji} , where

(Ax1) $x=u_i \wedge u=w_j \rightarrow x=w_{ji}$ (2·7·7 axioms).

The proof of $\varphi u_i \wedge u_i=w_j \rightarrow \varphi w_j$ is completely analogous, we omit it.



We shall often use the following abbreviation (because of the definition of $\circledcirc$): Let $u,w \in \{x,y,z\}$. Then

$$\Delta(u,w) \overset{d}{=} (u_0=w_{00} \wedge u_1=w_{11} \wedge w_{01}=w_{10}) .$$

Thus $\varphi \circledcirc \psi$ is $\exists y(\varphi y_0 \wedge \psi y_1 \wedge \Delta(x,y))$. Warning: When writing $\Delta(u,w)$ we do not use the substitution convention (S) (this is the only exception). I.e. $\Delta(z,y)$ is $z_0=y_{00} \wedge z_1=y_{11} \wedge y_{01}=y_{10}$ and not $\exists x(x=z \wedge \Delta(x,y))$!

Next we show that in the definition of $\varphi\odot\psi$ we can use z instead of y :

(2) $Ax \vdash_{\Gamma} \varphi\odot\psi \rightarrow \exists z(\varphi z_0 \wedge \psi z_1 \wedge \Delta(x,z))$.

For, $\exists y(\varphi y_0 \wedge \psi y_1 \wedge \Delta(x,y)) \vdash_{\Gamma}$ (by FO, CA $\models c_2X=c_2(d_{12}\cdot c_2X)$)

$\exists yz(y=z \wedge \varphi y_0 \wedge \psi y_1 \wedge \Delta(x,y)) \vdash_{\Gamma}$ (by (1), Ax2, CA)

$\exists yz(\varphi z_0 \wedge \psi z_1 \wedge \Delta(x,z)) \vdash_{\Gamma}$ (by FO)

$\exists z(\varphi z_0 \wedge \psi z_1 \wedge \Delta(x,z))$, where

(Ax2) $y=z \wedge \Delta(x,y)) \rightarrow \Delta(x,z)$.

In general, we can use y instead of z, or vice versa, as "auxiliary" bounded variable. We shall need this in the following concrete (special) form:

(3) $Ax \vdash_{\Gamma} \exists z(z=x \wedge \varphi z_0 \wedge \psi z_1) \rightarrow \exists y(y=x \wedge \varphi y_0 \wedge \psi y_1)$.

For, $\exists z(z=x \wedge \varphi z_0 \wedge \psi z_1) \vdash_{\Gamma}$ (by FO, CA)

$\exists yz(z=y \wedge z=x \wedge \varphi z_0 \wedge \psi z_1) \vdash_{\Gamma}$ (by CA, (1))

$\exists y(y=x \wedge \varphi y_0 \wedge \psi y_1)$.

Let $\{i,j\}=\{0,1\}$ (i.e. $i=0, j=1$ or $i=1, j=0$).

$$(4)\ (\varphi\odot\psi)y_i \rightarrow \exists z(\varphi z_{i0} \wedge \psi z_{i1} \wedge y_{i0}=z_{i00} \wedge z_{i01}=z_{i10} \wedge z_{i11}=y_{i1} \wedge$$
$$\wedge\, y_j=z_j) \quad ,$$

see Figure 4.

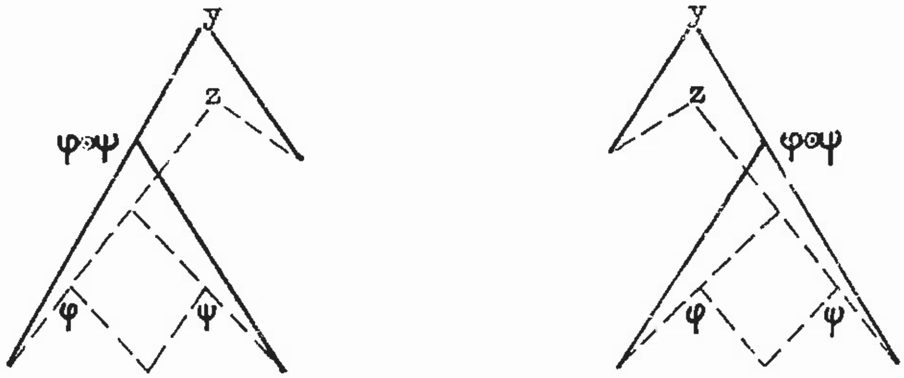


FIGURE 4

(Illustration of (4).)

For,

$\exists x(x=y_i \wedge \varphi\odot\psi) \vdash_r$ (by (2), FO)

$\exists xz(x=y_i \wedge \varphi z_0 \wedge \psi z_1 \wedge \Delta(x,z)) \vdash_r$ (by Ax3)

$\exists z(y_{i0}=z_{00} \wedge y_{i1}=z_{11} \wedge z_{01}=z_{10} \wedge \varphi z_0 \wedge \psi z_1) \vdash_r$ (by FO, CA)

$\exists zx(z=x \wedge y_{i0}=z_{00} \wedge y_{i1}=z_{11} \wedge z_{01}=z_{10} \wedge \varphi z_0 \wedge \psi z_1) \vdash_r$ (by Ax4, FO, CA)

$\exists x(y_{i0}=x_{00} \wedge y_{i1}=x_{11} \wedge x_{01}=x_{10} \wedge \exists z(z=x \wedge \varphi z_0 \wedge \psi z_1)) \vdash_r$ (by (3))

$\exists x(y_{i0}=x_{00} \wedge y_{i1}=x_{11} \wedge x_{01}=x_{10} \wedge \exists y(y=x \wedge \varphi y_0 \wedge \psi y_1)) \vdash_r$ (by Ax5)

$\exists xz(y_j=z_j \wedge z_i=x \wedge y_{i0}=x_{00} \wedge y_{i1}=x_{11} \wedge x_{01}=x_{10} \wedge \exists y(y=x \wedge \varphi y_0 \wedge \psi y_1)) \vdash_r$
(by Ax6)

$\exists z(y_{i0}=z_{i00} \wedge z_{i11}=y_{i1} \wedge z_{i01}=z_{i10} \wedge y_j=z_j \wedge \exists xy(z_i=x \wedge y=x \wedge \varphi y_0 \wedge \psi y_1)) \vdash_r$
(by Ax7, (1))

$\exists z(y_{i0}=z_{i00}\wedge z_{i11}=y_{i1}\wedge z_{i01}=z_{i10}\wedge y_j=z_j\wedge\exists xy(\varphi z_{i0}\wedge\psi z_{i1}))\ \vdash_r$ (by FO)

$\exists z(\varphi z_{i0}\wedge\psi z_{i1}\wedge y_{i0}=z_{i00}\wedge z_{i01}=z_{i10}\wedge z_{i11}=y_{i1}\wedge y_j=z_j)$, where

(Ax3) $(x=y_i\wedge\Delta(x,z)) \rightarrow (y_{i0}=z_{00}\wedge y_{i1}=z_{11}\wedge z_{01}=z_{10})$ (two axioms)

(Ax4) $(z=x\wedge y_{i0}=z_{00}\wedge y_{i1}=z_{11}\wedge z_{01}=z_{10}) \rightarrow (y_{i0}=x_{00}\wedge y_{i1}=x_{11}\wedge x_{01}=x_{10})$

(Ax5) $\forall xy\exists z(y_j=z_j\wedge z_i=x)$

(Ax6) $(z_i=x\wedge y_{i0}=x_{00}\wedge y_{i1}=x_{11}\wedge x_{01}=x_{10}) \rightarrow (y_{i0}=z_{i00}\wedge z_{i11}=y_{i1}\wedge$
$\wedge z_{i01}=z_{i10})$

(Ax7) $z_i=x\wedge y=x \rightarrow y=z_i$.

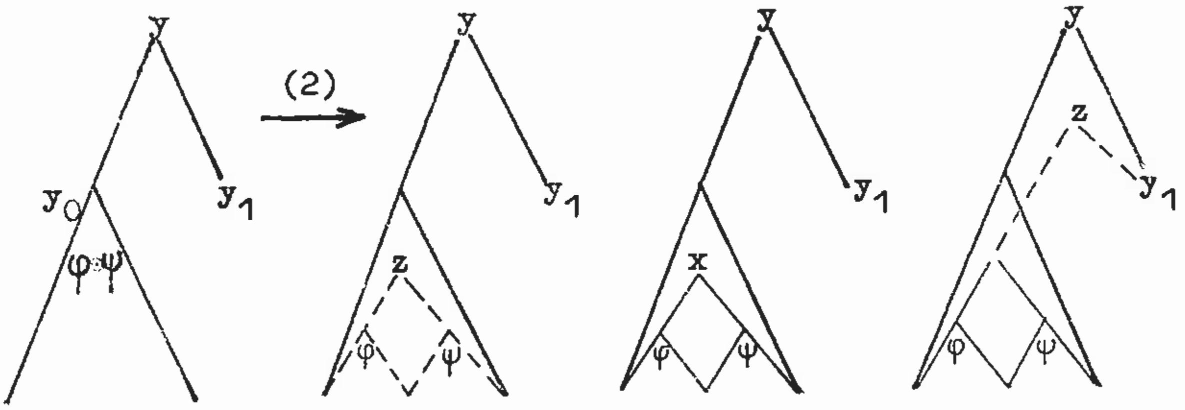


FIGURE 5

(The idea of the proof of (4)).

In what follows, we won't always write out the axioms of Ax explicitly.

(5) Ax $\vdash_{r}$ $(\varphi\odot\psi)\odot\gamma \leftrightarrow \varphi\odot(\psi\odot\gamma)$.

For,

$(\varphi\odot\psi)\odot\gamma$ $\vdash_{r}$ (by definition)

$\exists y((\varphi\odot\psi)y_0 \wedge \gamma y_1 \wedge \Delta(x,y))$ $\vdash_{r}$ (by (4), FO)

$\exists yz(\varphi z_{00} \wedge \psi z_{01} \wedge \gamma y_1 \wedge y_{00}=z_{000} \wedge z_{001}=z_{010} \wedge z_{011}=y_{01} \wedge y_1=z_1 \wedge \Delta(x,y))$ $\vdash_{r}$

(by (1), Ax)

$\exists z(\varphi z_{00} \wedge \psi z_{01} \wedge \gamma z_1 \wedge x_0=z_{000} \wedge x_1=z_{11} \wedge z_{001}=z_{010} \wedge z_{011}=z_{10})$ $\vdash_{r}$ (by Ax8)

$\exists zy(\varphi z_{00} \wedge \psi z_{01} \wedge \gamma z_1 \wedge z_{00}=y_0 \wedge \exists x(x=y_1 \wedge \exists y[z_{01}=y_0 \wedge z_1=y_1 \wedge \Delta(xy)]) \wedge \Delta(xy))$

$\vdash_{r}$ (by (1),FO)

$\exists y(\varphi y_0 \wedge \exists x(x=y_1 \wedge \exists y[\psi y_0 \wedge \gamma y_1 \wedge \Delta(xy)]) \wedge \Delta(x,y))$ $\vdash_{r}$ (by definition)

$\underline{\varphi\odot(\psi\odot\gamma)}$ $\vdash_{r}$ (by definition)

$\exists y(\varphi y_0 \wedge (\psi\odot\gamma)y_1 \wedge \Delta(x,y))$ $\vdash_{r}$ (by (4), FO)

$\exists yz(\varphi y_0 \wedge \psi z_{10} \wedge \gamma z_{11} \wedge y_{10}=z_{100} \wedge z_{101}=z_{110} \wedge z_{111}=y_{11} \wedge y_0=z_0 \wedge \Delta(xy))$ $\vdash_{r}$

(by (1),Ax,FO)

$\exists z(\varphi z_0 \wedge \psi z_{10} \wedge \gamma z_{11} \wedge x_1=z_{111} \wedge x_0=z_{00} \wedge z_{110}=z_{101} \wedge z_{100}=z_{01})$ $\vdash_{r}$

(by Ax8, FO)

$\exists zy(\varphi z_0 \wedge \psi z_{10} \wedge \gamma z_{11} \wedge z_{11}=y_1 \wedge \exists x(x=y_0 \wedge \exists y(z_{10}=y_1 \wedge z_0=y_0 \wedge \Delta(xy))) \wedge \Delta(xy))$

$\vdash_{r}$ (by FO, (1))

$\exists y(\gamma y_1 \wedge \exists x(x=y_0 \wedge \exists y(\psi y_1 \wedge \varphi y_0 \wedge \Delta(xy))) \wedge \Delta(x,y))$ $\vdash_{r}$ (by definition)

$(\varphi\odot\psi)\odot\gamma$.

(Ax8) $x_i = z_{iii} \wedge x_j = z_{jj} \wedge z_{iij} = z_{iji} \wedge z_{ijj} = z_{ji} \quad \rightarrow$

$\exists y(z_{ii} = y_i \wedge \exists x(x = y_j \wedge \exists y[z_{ij} = y_i \wedge z_j = y_j \wedge \Delta(xy)]) \wedge \Delta(xy))$.

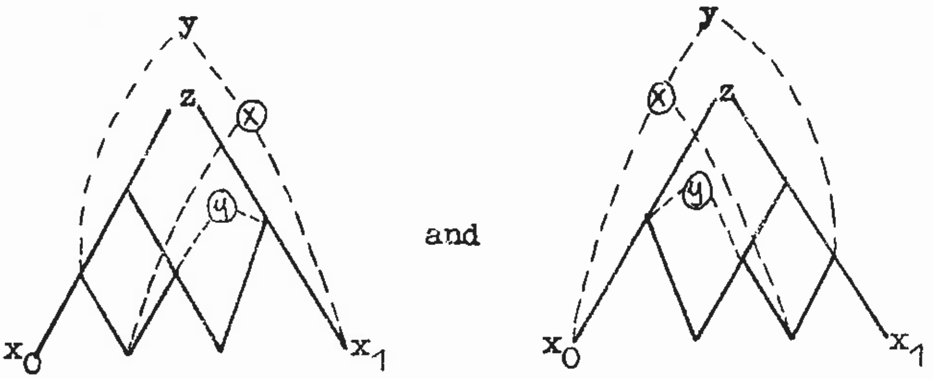


FIGURE 6

(Illustration of Ax8).

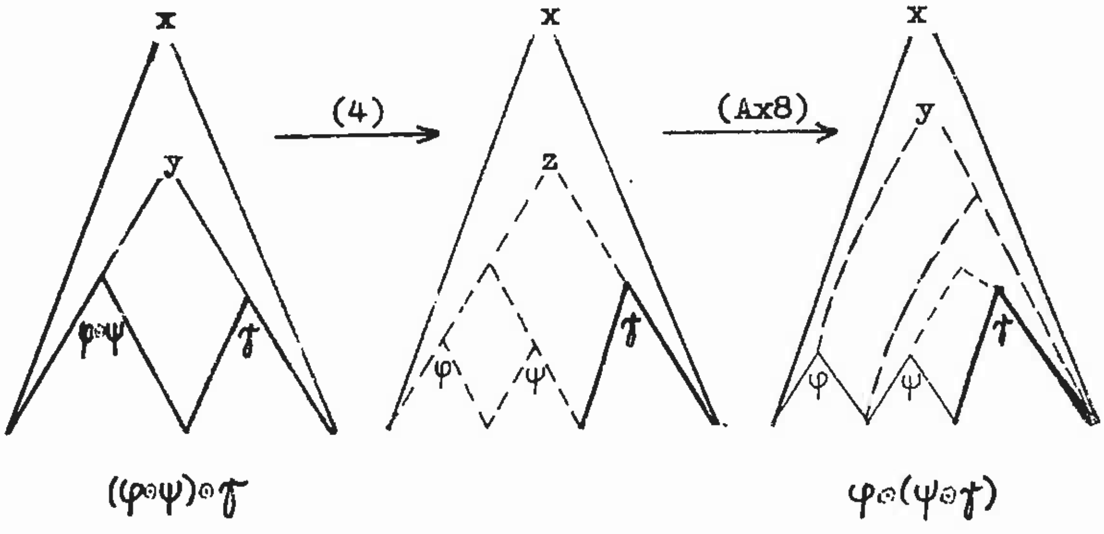


FIGURE 7

(Jllustration of the proof of (5)).

Now we turn to proving condition (3) of [Ma78a] i.e. we prove that $1'$ is the identity of $\odot$ in $\mathfrak{R}$.

Let $u \in \{y,z\}$ and $i \in 2^*$, $|i| \leq 2$.

(6) $Ax \vdash_r \varphi u_i \wedge x=u_i \rightarrow \varphi$.

For, let $w \in \{y,z\}$, $w \neq u$.

$\varphi u_i \wedge x=u_i \vdash_r$ (by FO, CA)

$\exists w(w=x \wedge x=u_i \wedge \varphi u_i) \vdash_r$ (by Ax)

$\exists w(w=x \wedge w=u_i \wedge \varphi u_i) \vdash_r$ (by FO, definition)

$\exists w(w=x \wedge \exists x(w=u_i \wedge x=u_i \wedge \varphi)) \vdash_r$ (by Ax)

$\exists w(w=x \wedge \exists x(x=w \wedge \varphi)) \vdash_r$ (by CA $\models s_0^2 s_2^0 c_2 X = c_2 X$, FO)

φ.

(7) $Ax \vdash_r \varphi \odot 1' \leftrightarrow \varphi$.

For, $\varphi \odot 1' \vdash_r$ (by definition)

$\exists y(\varphi y_0 \wedge 1' y_1 \wedge \Delta(x,y)) \vdash_r$ (by Ax9)*/

$\exists y(\varphi y_0 \wedge x=y_0) \vdash_r$ (by (6), FO)

$\underline{\varphi} \vdash_r$ (by $\varphi \in Ora$)

$\varphi \wedge pair(x) \vdash_r$ (by Ax10, FO)

$\exists y(\varphi \wedge x=y_0 \wedge 1' y_1 \wedge \Delta(x,y)) \vdash_r$ (by CA)

$\exists y(\exists x(x=y_0 \wedge \varphi) \wedge 1' y_1 \wedge \Delta(x,y)) \vdash_r$ (by definition)

$\varphi \odot 1'$ where

---

*/ Here we needed the stronger π' (uniqueness of the pair), i.e. $\pi \not\models$ (Ax9) while $\pi' \models$ (Ax9).

(Ax9) $1'y_i \wedge \Delta(x,y) \to x=y_j$ for $\{i,j\}=\{0,1\}$, and

(Ax10) $\mathrm{pair}(x) \to \exists y(x=y_i \wedge 1'y_j \wedge \Delta(x,y))$ for $\{i,j\}=\{0,1\}$.

FIGURE 8

(Illustration of (Ax10)).

(8) Ax $\vdash_{r} 1' \odot \varphi \leftrightarrow \varphi$.

For, $1' \odot \varphi \vdash_{r}$ (by definition)

$\exists y(1'y_0 \wedge \varphi y_1 \wedge \Delta(x,y)) \vdash_{r}$ (by Ax9)

$\exists y(\varphi y_1 \wedge x=y_1) \vdash_{r}$ (by (6), FO)

$\varphi \vdash_{r}$ (by $\varphi \in$ Ora)

$\varphi \wedge \mathrm{pair}(x) \vdash_{r}$ (by Ax10, FO)

$\exists y(\varphi \wedge x=y_1 \wedge 1'y_0 \wedge \Delta(x,y)) \vdash_{r}$ (by CA, FO, definition)

$1' \odot \varphi$.

Our last condition to prove, condition (4) in [Ma78a] is $\varphi\cdot(\psi;\gamma)=0$ iff $\psi\cdot(\varphi;\gamma^{\smile})=0$ iff $\gamma\cdot(\psi^{\smile};\varphi)=0$, for every $\varphi,\psi,\gamma \in R$.

Let $\xi,\eta \in$ Ora be arbitrary. To prove $\xi/{\equiv_{Ax}}=0 \Rightarrow \eta/{\equiv_{Ax}}=0$ it is enough to prove

(*) Ax $\vdash_{r} \exists x\eta \to \exists x\xi$

because of the following. Assume that $\xi/{\equiv_{Ax}}=0$ in $\mathcal{R}$.

This means that $Ax \vdash_r (\xi \leftrightarrow \underline{F})$ (where $\underline{F}$ is the "false" formula). Now,

$(\xi \leftrightarrow \underline{F}) \vdash_r$ (by CA)

$\exists x\xi \leftrightarrow \underline{F} \vdash_r$ (by BA)

$\neg\exists x\xi \vdash_r$ (by (*))

$\neg\exists x\eta \vdash_r$ (by BA)

$(\exists x\eta \leftrightarrow \underline{F}) \vdash_r$ (by CA)

$(\eta \leftrightarrow \underline{F})$,

that is, $\eta/{\equiv_{Ax}} = 0$.

To prove condition (4), first we prove some (natural) auxiliary lemmas.

(9) $Ax \vdash_r (\varphi z_1 \wedge y_{10}=z_{11} \wedge y_{11}=z_{10}) \rightarrow \varphi^{\smile} y_1$.

For,

$\varphi z_1 \wedge y_{10}=z_{11} \wedge y_{11}=z_{10} \vdash_r$ (by FO, Ax)

$\exists x(x=y_1 \wedge \varphi z_1 \wedge \exists y(z_1=y \wedge x_0=y_1 \wedge x_1=y_0)) \vdash_r$ (by FO, (1), def.)

$\varphi^{\smile} y_1$.

(10) $Ax \vdash_r (\varphi \wedge y_{00}=x_1 \wedge y_{01}=x_0) \rightarrow \varphi^{\smile} y_0$.

For,

$\exists z(x=z \wedge \varphi \wedge y_{00}=x_1 \wedge y_{01}=x_0) \vdash_r$ (by CA $\models d_{02} \cdot X \leq c_0(d_{02} \cdot X)$, Ax)

$\exists z(\varphi z \wedge y_{00}=z_1 \wedge y_{01}=z_0) \vdash_r$ (by FO, Ax)

$\exists x(x=y_0 \wedge \exists z(\varphi z \wedge x_0=z_1 \wedge x_1=z_0)) \vdash_r$ (by FO, CA, (1), Ax, cf. the proof of (3))

$\exists x(x=y_0 \wedge \exists y(\varphi y \wedge x_0=y_1 \wedge x_1=y_0)) \vdash_{r}$ (by definition)

$\varphi^{\smile} y_0$.

(11) $Ax \vdash_{r} \varphi^{\smile} z_i \rightarrow \exists y(\varphi y \wedge y_0=z_{i1} \wedge y_1=z_{i0})$ for $i \in \{0,1\}$.

For, $\varphi^{\smile} z_i \vdash_{r}$ (by definition)

$\exists x(x=z_i \wedge \exists y(\varphi y \wedge x_0=y_1 \wedge x_1=y_0)) \vdash_{r}$ (by FO, Ax)

$\exists y(\varphi y \wedge y_0=z_{i1} \wedge y_1=z_{i0})$.

(12) $Ax \vdash_{r} (\varphi^{\smile} z_1 \wedge x_0=z_{11} \wedge x_1=z_{10}) \rightarrow \varphi$.

For, $\varphi^{\smile} z_1 \wedge x_0=z_{11} \wedge x_1=z_{10} \vdash_{r}$ (by (11), FO)

$\exists y(\varphi y \wedge y_0=z_{11} \wedge y_1=z_{10} \wedge x_0=z_{11} \wedge x_1=z_{10}) \vdash_{r}$ (by Ax)

$\exists y(\varphi y \wedge x=y) \vdash_{r}$ (by (6))

φ.

(13) $Ax \vdash_{r} (\varphi^{\smile} z_0 \wedge y_{00}=z_{01} \wedge y_{01}=z_{00}) \rightarrow \varphi y_0$.

For, $\varphi^{\smile} z_0 \wedge y_{00}=z_{01} \wedge y_{01}=z_{00} \vdash_{r}$ (by Ax, FO, def.)
$\exists x(x=y_0 \wedge x_0=z_{01} \wedge x_1=z_{00} \wedge \exists x(x=z_0 \wedge \exists y[\varphi y \wedge x_0=y_1 \wedge x_1=y_0])) \vdash_{r}$ (by Ax, FO)
$\exists x(x=y_0 \wedge x_0=z_{01} \wedge x_1=z_{00} \wedge \exists y[\varphi y \wedge y_0=z_{01} \wedge y_1=z_{00}]) \vdash_{r}$ (by FO, Ax, (11))

$\exists x(x=y_0 \wedge \exists y[\varphi y \wedge x=y]) \vdash_{r}$ (by (6), definition)

φy_0.

(14) $Ax \vdash_{r} \exists x[\varphi \wedge (\psi \odot \gamma)] \rightarrow \exists x[\psi \wedge (\varphi \odot \gamma^{\smile})]$ and

$Ax \vdash_{r} \exists x[\psi \wedge (\varphi \odot \gamma^{\smile})] \rightarrow \exists x[\gamma \wedge (\psi^{\smile} \odot \varphi)]$ and

$Ax \vdash_{r} \exists x[\gamma \wedge (\psi^{\smile} \odot \varphi)] \rightarrow \exists x[\varphi \wedge (\psi \odot \gamma)]$.

For,

$\exists x[\varphi\wedge(\psi\odot\gamma)]$ $\vdash_{r}$ (by (2), FO)

$\exists xz[\varphi\wedge\psi z_0\wedge\gamma z_1\wedge\Delta(x,z)]$ $\vdash_{r}$ (by Ax11, FO)

$\exists xyz[\varphi\wedge\psi z_0\wedge\gamma z_1\wedge x=y_0\wedge y_{10}=z_{11}\wedge y_{11}=z_{10}\wedge\exists x(x=z_0\wedge\Delta(xy))]$ $\vdash_{r}$?
(by CA, (6), (9), Ax)

$\exists y[\varphi y_0\wedge\gamma^{\smile}y_1\wedge\exists x(\psi\wedge\Delta(x,y))]$ $\vdash_{r}$ (by FO)

$\exists x[\psi\wedge\exists y(\varphi y_0\wedge\gamma^{\smile}y_1\wedge\Delta(x,y))]$ $\vdash_{r}$ (by definition)

$\underline{\exists x[\psi\wedge(\varphi\odot\gamma^{\smile})]}$ $\vdash_{r}$ (by (2), FO)

$\exists xz[\psi\wedge\varphi z_0\wedge\gamma^{\smile}z_1\wedge\Delta(x,z)]$ $\vdash_{r}$ (by Ax12, FO)

$\exists xyz[\psi\wedge\varphi z_0\wedge\gamma^{\smile}z_1\wedge z_0=y_1\wedge y_{00}=x_1\wedge y_{01}=x_0\wedge\exists x(x_0=z_{11}\wedge x_1=z_{10}\wedge\Delta(xy))]$
$\vdash_{r}$ (by (10), (1), FO)

$\exists y[\psi^{\smile}y_0\wedge\varphi y_1\wedge\exists x(\gamma^{\smile}z_1\wedge x_0=z_{11}\wedge x_1=z_{10}\wedge\Delta(x,y))]$ $\vdash_{r}$
(by (12), FO, def.)

$\underline{\exists x[\gamma\wedge(\psi^{\smile}\odot\varphi)]}$ $\vdash_{r}$ (by (2), FO)

$\exists xz[\gamma\wedge\psi^{\smile}z_0\wedge\varphi z_1\wedge\Delta(x,z)]$ $\vdash_{r}$ (by Ax11, FO)

$\exists xyz[\gamma\wedge\psi^{\smile}z_0\wedge\varphi z_1\wedge x=y_1\wedge y_{00}=z_{01}\wedge y_{01}=z_{00}\wedge\exists x(x=z_1\wedge\Delta(xy))]$ $\vdash_{r}$
(by CA, (13), (6))

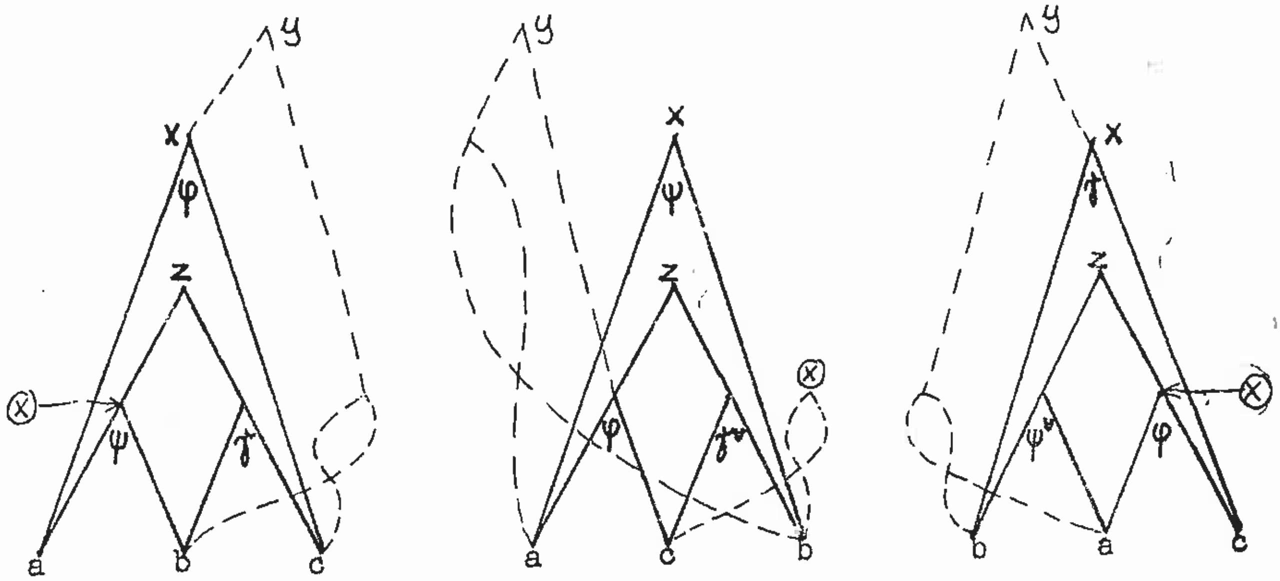
$\exists y[\psi y_0\wedge\gamma y_1\wedge\exists x(\varphi\wedge\Delta(x,y))]$ $\vdash_{r}$ (by FO, CA, definition)

$\exists x[\varphi\wedge(\psi\odot\gamma)]$, where

(Ax11) $\Delta(x,z) \rightarrow \exists y(x=y_i \wedge y_{j0}=z_{j1} \wedge y_{j1}=z_{j0} \wedge \exists x[x=z_i \wedge \Delta(x,y)])$,

for*/ $\{i,j\}=\{0,1\}$.

(Ax12) $\Delta(x,z) \rightarrow \exists y(z_0=y_1 \wedge y_{00}=x_1 \wedge y_{01}=x_0 \wedge \exists x[x_0=z_{11} \wedge x_1=z_{10} \wedge \Delta(xy)])$.



(Ax11)
(i=0, j=1)

(Ax12)

(Ax11)
(i=1, j=0)

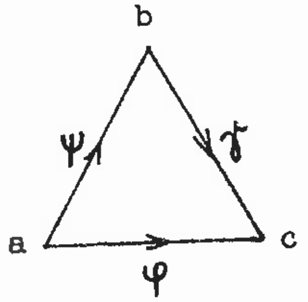


FIGURE 9

(Illustration of the proof of (13), and of Ax11, 12).

QED(Proposition 2.10.)

Now we are ready to prove Theorems 1-2.

Proof of Thm.1(a): We prove more: Let $\Lambda=\langle\alpha,R,\rho\rangle$ be any nonmonadic language with $\alpha\geqslant 3$. We will show that Λ has G.i. Since Λ is nonmonadic, there is $i\in\beta$ $=\mathrm{Do}R$ such that $\rho_i\geqslant 2$. First we assume that $R_0=E$ and $\rho_0=2$. Later we can repeat the proof with writing $R_i(v_0\dots v_{\rho_i-1})$ everywhere in place of $E(v_0,v_1)$ and conjuncting the formula $R_i(v_0\dots v_{\rho_i-1})\leftrightarrow\exists v_2\dots\exists v_{\rho_i-1}R_i(v_0\dots v_{\rho_i-1})$ to ψ below.

Let $p_i(x,y)\in Fm_3^2$ $(i\in 2)$ and $\lambda\in Fm_\omega^0$ be fixed such that λ is inseparable and semantically consistent with π'. (Such λ and $p_i(x,y)$ exist by Lemma2.7.) Let $\mathfrak{G}\overset{d}{=}$ $\mathfrak{Fr}_1\mathrm{SimRA}$ and let $h:\mathfrak{G}\to\mathfrak{Ora}$ be the homomorphism taking the free generator of $\mathfrak{G}$ to $\bar{e}\overset{d}{=}\mathrm{pair}(x)\wedge E(x_0x_1)$ where $E(x_0x_1)\overset{d}{=}\exists y\exists z(z=x_0\wedge y=x_1\wedge E(z,y))$. Then $h:G\to Fm^\Lambda$, too, since $\mathrm{Ora}\subseteq Fm^\Lambda$. Recall that $\eta=rf\lambda\cdot\pi_{RA}\in G$ and $\mathrm{pair}(x)\overset{d}{=}\exists yp_0(x,y)\wedge\exists yp_1(x,y)$. Let $\psi\overset{d}{=}\wedge Ax\wedge\forall x(\mathrm{pair}(x)\leftrightarrow h\eta)\wedge\exists x\mathrm{pair}(x)$. Below we will show that $\psi\in Fm^\Lambda$ is $\vdash_{r,\Lambda}$ consistent and that ψ cannot be extended to a decidable, complete theory.

CLAIM 1. $\nvdash_{r,\Lambda}\neg\psi$.

Proof. It is enough to show that $\mathfrak{M}\models\psi$ for some model, since then by the soundness of the proof system $\vdash_{r,\Lambda}$ we will have $\nvdash_{r,\Lambda}\neg\psi$. By our choice of λ and $p_i(x,y)$ $(i\in 2)$ there is a model $\mathfrak{M}$ such that $\mathfrak{M}\models\lambda\wedge\pi'$. Then

$\mathfrak{M} \models \lambda \wedge \pi \wedge Ax$ by $\pi' \models \pi \wedge Ax$. We will show that $\mathfrak{M} \models \forall x(\mathrm{pair}(x) \leftrightarrow h\eta) \wedge \exists x \mathrm{pair}(x)$. Let Pair $\overset{d}{=} \{a \in M : \mathfrak{M} \models \mathrm{pair}(x)[a]\}$. For every $a \in \mathrm{Pair}$ let a_0, a_1 denote a's p_0 - and p_1-image in $\mathfrak{M}$ (i.e. its first and second "coordinates"). For every $Z \subseteq {}^2M$ define $\mathbf{Z} \overset{d}{=} \{a \in \mathrm{Pair} : (a_0, a_1) \in Z\}$. Let $\mathfrak{R}$ be the full relational set algebra with base M (i.e. $\mathfrak{R} \overset{d}{=} \mathcal{R}(M)$ in the notation of [J82]) and let $\mathfrak{M} \overset{d}{=} \langle M, \in \rangle$. Finally, for every $\varphi \in Fm_3^1$ define $\varphi^{\mathfrak{M}} \overset{d}{=} \{a \in M : \mathfrak{M} \models \varphi[a]\}$. Now, using the definition of the operations in $\mathfrak{Ora}$ one can prove by induction that

$$(*) \qquad (h\tau)^{\mathfrak{M}} = \overline{\tau^{\mathfrak{R}}(\varepsilon)} \qquad \text{for every} \quad \tau \in RAT.$$

By $\mathfrak{M} \models \pi$, $\mathfrak{M} \models \lambda$ we have $\mathfrak{M} \models f\lambda$ (cf. Lemma 2.2), hence $(rf\lambda)^{\mathfrak{R}}(\varepsilon) = {}^2M$ by Lemma 2.3, i.e. $\overline{(rf\lambda)^{\mathfrak{R}}(\varepsilon)} =$ Pair. By (*) then $(hrf\lambda)^{\mathfrak{M}} =$ Pair. Let $p \overset{d}{=} r(p_0(x,y))$ and $q \overset{d}{=} r(p_1(x,y))$. Then by Lemma 2.3 $\{a \in {}^2M : \mathfrak{M} \models p_0(x,y)[a]\} = p^{\mathfrak{R}}(\varepsilon)$ and $\{a \in {}^2M : \mathfrak{M} \models p_1(x,y)[a]\} = q^{\mathfrak{R}}(\varepsilon)$. Then by $\mathfrak{M} \models \pi$ and by the definition of π_{RA} we have $\pi_{RA}^{\mathfrak{R}}(\varepsilon) = {}^2M$, hence by (*) again $(h\pi_{RA})^{\mathfrak{M}} =$ Pair. Now, $h\eta = h(rf\lambda \cdot \pi_{RA}) = h(rf\lambda) \wedge h(\pi_{RA})$, hence $(h\eta)^{\mathfrak{M}} =$ Pair (by the above). This means that $\mathfrak{M} \models \forall x(\mathrm{pair}(x) \leftrightarrow h\eta)$. Clearly, $\mathfrak{M} \models \exists x \mathrm{pair}(x)$ by $\mathfrak{M} \models \pi$. QED(Claim 1)

CLAIM 2. ψ cannot be extended to a decidable, complete theory.

Proof. Assume $\psi \in T = \hat{T} \overset{d}{=} \{\varphi \in Fm^{\Lambda} : T \vdash_{r,\Lambda} \varphi\}$ and T is decidable and complete. Define

$$R \overset{d}{=} \{(\tau, \sigma) \in {}^2G : (h\tau \leftrightarrow h\sigma) \in T\}.$$

We will show that R is a decidable congruence of $\mathfrak{G}$ taking η to 1 such that factor $\mathfrak{G}/R$ is a nontrivial RA; these contradict Lemma 2.6.

(1) R is a congruence on $\mathfrak{G}$ since $T = \hat{T}$

can be seen as follows. Define $S \overset{d}{=} \{(\varphi, \varrho) \in {}^2Fm^{\wedge} :$ $(\varphi \leftrightarrow \psi) \in T\}$. It is enough to show that S is a congruence on $\mathfrak{Fm}^{\wedge}$. S is an equivalence relation and is a congruence w.r.t. the Boolean operations $\vee, \wedge, \neg$ by $(\!(1)\!) \in \Lambda_T^{\wedge}$ and (MP). Let $i \in \alpha$. That S is a congruence w.r.t. $\exists v_i$ can be seen as follows. Assume $(\varphi \leftrightarrow \varrho) \in T$. Then, by $(\!(1)\!)$, (MP), $(\neg\varphi \to \neg\varrho) \in T$, then by (G), $\forall v_i(\neg\varphi \to \neg\varrho) \in T$, then by $(\!(2)\!)$, $(\forall v_i \neg\varphi \to \forall v_i \neg\varrho) \in T$, then by $(\!(1)\!)$, $(\neg\forall v_i \neg\varrho \to \neg\forall v_i \neg\varphi) \in T$, hence by $(\!(9)\!)$, $(\exists v_i \varrho \to \exists v_i \varphi) \in T$. The other direction, $(\exists v_i \varphi \to \exists v_i \varrho) \in T$ can be seen analogously. Thus S is a congruence on $\mathfrak{Fm}^{\wedge}$, hence R is a congruence on $\mathfrak{G}$ (since $\mathfrak{Cra}$ is a generalized reduct of $\mathfrak{Fm}^{\wedge}$).

Clearly,

(2) R is decidable since T is decidable.

(3) $\mathfrak{G}/R \in RA$ by $Ax \subseteq T$ and Proposition 2.10,

can be seen as follows. Let $\tau, \sigma \in G$. Assume $(h\tau, h\sigma) \in \equiv_{Ax}$. Then $Ax \vdash_{r,\Lambda} h\tau \leftrightarrow h\sigma$, hence $T \vdash_{r,\Lambda} h\tau \leftrightarrow h\sigma$ by $Ax \subseteq T$, therefore $(h\tau \leftrightarrow h\sigma) \in T$ by $\hat{T} = T$. Therefore $(\tau, \sigma) \in R$. We have seen that $(h\tau, h\sigma) \in \equiv_{Ax}$ implies $(\tau, \sigma) \in R$. Thus $\mathfrak{G}/R$ is a homomorphic image of $h^{*}\mathfrak{G}/\equiv_{Ax} \subseteq \mathfrak{Cra}/\equiv_{Ax}$ which is a RA by Prop.2.10. Hence $\mathfrak{G}/R \in RA$, too.

(4) $\eta \in 1/R$ by $(\mathrm{pair}(x) \leftrightarrow h\eta) \in T$,

since $h1 = \mathrm{pair}(x)$.

(5) $1/R \neq 0/R$ by $\exists x\,\mathrm{pair}(x) \in T$,

since $(\mathrm{pair}(x) \leftrightarrow \underline{F}) \in T$ would imply $\forall x \neg \mathrm{pair}(x) \in T$ (by ⟪1⟫, (MP),(G)).

Now (1)-(5) above contradict Lemma2.6. Therefore ψ cannot be extended to a complete and decidable theory.
<u>QED(Claim 2)</u>

Claims 1,2 show that Λ has Gödel's incompleteness.
<u>QED(Theorem 1(a))</u>

Theorem 1(b) follows from Thm.2(a). <u>QED(Theorem 1)</u>

<u>Proof of Thm.2(a):</u> Let $\beta \geq 1$, $3 \leq \alpha < \omega$, $K \subseteq CA_\alpha$ and $\Delta : \beta \to (\alpha+1)$ be such that $\overline{Eq}K$ is recursively enumerable (r.e. from now on), $\mathfrak{Rf}(\kappa,\alpha) \in K$ for some infinite κ and $\Delta(j) \geq 2$ for some $j \in \beta$. We will show that $\mathfrak{Zd}\,\mathfrak{Fr}_\beta^{(\Delta)}K$ is not atomic. If $\beta \geq \omega$ then "$\mathfrak{Zd}\,\mathfrak{Fr}_\beta^{(\Delta)}K$ is atomless" can be proved analogously to [HMT]2.5.13 (the only change is that we use $c_{(\alpha)}g_\eta$ instead of g_η in that proof). Therefore we may assume that $\beta < \omega$. Hence Δ is r.e. Let $\Lambda = \langle \alpha, R, \Delta \rangle$ be a language. By [HMT]4.3.1, 4.3.25, 4.3.56 there is a recursive $\xi : \mathfrak{Fr}_\beta CA_\alpha \Rightarrow Fm^\Lambda$ such that ξ induces an isomorphism between $\mathfrak{Fr}_\beta^{(\Delta)}CA_\alpha$ and $\mathfrak{Fm}^\Lambda/\equiv_0$, and in addition, for any $\tau \in Fr_\beta CA_\alpha$ and model $\mathfrak{M}$ for Λ if $\mathfrak{Cs}^{\mathfrak{M}} \models \tau = 1[m]$ then $\mathfrak{M} \models \xi\tau$, where $(\forall i \in \beta)m(g_i) = R_i^{\mathfrak{M}}$ and $g_i (i \in \beta)$ are

the standard free generators of $\mathfrak{Fr}_\beta CA_\alpha$. Define $\Sigma \overset{d}{=} \{\xi\tau : \tau \in 1/Cr_\beta^{(\Delta)}K\}$. Then Σ is r.e. and $\mathfrak{Fr}_\beta^{(\Delta)}K \cong \mathfrak{Fm}^\Lambda/{\equiv_\Sigma}$. Let λ be an inseparable formula such that $\lambda\wedge\pi'$ has a model (in Λ). Such $\lambda\wedge\pi'$ exists by Lemma 2.7 and since $(\exists i\in\beta)\,\Delta(i)\geqslant 2$. Let $\psi \overset{d}{=} \forall v_0\ldots v_{\alpha-1}(Ax\wedge\forall x(pair(x)\leftrightarrow h\eta)\wedge\exists x\,pair(x))$ as in the proof of Thm.1(a). Then $\psi/{\equiv_\Sigma}\in \mathfrak{Zd}\,\mathfrak{F}$ where $\mathfrak{F}\overset{d}{=}\mathfrak{Fm}^\Lambda/{\equiv_\Sigma}$. First we show that $\psi/{\equiv_\Sigma}\neq 0$. Let $\mathfrak{M}=\langle \mathcal{K}, R_i^{\mathfrak{M}}\rangle_{i\in\beta}$ be a model of $\lambda\wedge\pi'$. Then $\mathfrak{M}\models\psi$ is shown in the proof of Thm.1(a). It remains to show that $\mathfrak{M}\models\Sigma$. Let $(\forall i\in\beta)\,m(g_i)\overset{d}{=}R_i^{\mathfrak{M}}$. Then $m : Fr_\beta CA_\alpha\to\mathfrak{Cs}^{\mathfrak{M}}$ satisfies Δ and $\mathfrak{Cs}^{\mathfrak{M}}\subseteq\mathfrak{Rf}(\kappa,\alpha)\in K$. Hence $\mathfrak{Cs}^{\mathfrak{M}}\models\tau=1$ for all $\tau\in 1/Cr_\beta^{(\Delta)}K$, therefore $\mathfrak{M}\models\xi\tau$ for all $\tau\in 1/Cr_\beta^{(\Delta)}K$ i.e. $\mathfrak{M}\models\Sigma$.

Assume that there is $\gamma\in Fm^{\Lambda,0}$ such that $\gamma/{\equiv_\Sigma}$ is an atom in $\mathfrak{F}$ below $\psi/{\equiv_\Sigma}$. Define

$$R \overset{d}{=} \{(\tau,\sigma)\in {}^2G \;:\; \gamma/{\equiv_\Sigma}\leqslant (h\tau\leftrightarrow h\sigma)/{\equiv_\Sigma}\}.$$

Then R is decidable since Σ is r.e. and $\gamma/{\equiv_\Sigma}$ is an atom. Now $\eta\in 1/R\neq 0/R$, $R\in Co\mathfrak{G}$ and $\mathfrak{G}/R\in RA$ can be seen exactly as in the proof of Thm.1(a). QED(Thm.2(a))

Proof of Thm.2(b): First we give a proof using Thm.1(a). Then we give a separate, direct proof for the case $K\subseteq RA$. Let $K\subseteq SA$ be such that $\overline{Eq}K$ is r.e. and $\mathfrak{R}(U)\in K$ for some infinite U. We may assume that K is a variety. Define $K'\overset{d}{=}\{\mathfrak{C}\in CA_3 : \mathfrak{Ra}\,\mathfrak{C}\in K\}$. Then $K'\subseteq CA_3$ is a variety and $\overline{Eq}K'$ is r.e. By $\mathfrak{R}(U)\in K$ we have that $\mathfrak{Rf}(|U|,3)\in K'$. Let $\beta\geqslant 1$ and let $\Delta:\beta\to 3$ be such that $(\forall i\in\beta)\Delta(i)=2$. Then $\mathfrak{Zd}\,\mathfrak{Fr}_\beta^{(\Delta)}K'$ is not atomic, by Thm.2(a). We will show, by using results from Maddux [Ma78], that $\mathfrak{Ra}\,\mathfrak{Fr}_\beta^{(\Delta)}K' \cong \mathfrak{Fr}_\beta K$. Let $\mathfrak{F} \overset{d}{=} \mathfrak{Fr}_\beta^{(\Delta)}K'$. Let G be the set of standard free generators of $\mathfrak{F}$. Then $G\subseteq Nr_2\mathfrak{F}$ by the definition of Δ. The class $CA_3'\subseteq CA_3$ is defined in [Ma78]p.127. Now $K'\subseteq CA_3'$ by that definition because $K\subseteq SA$. Hence $\mathfrak{F}\in CA_3'$, too. Then $Sg^{(\mathfrak{Ra}\,\mathfrak{F})}G = Ra\ Sg^{(\mathfrak{F})}G = Ra\,\mathfrak{F}$ by [Ma78]Thm(7),p.133. Thus G generates $\mathfrak{Ra}\,\mathfrak{F}$. We will show that every mapping from G into an element of K can be extended to a homomorphism from $\mathfrak{Ra}\,\mathfrak{F}$. This will show that $\mathfrak{Ra}\,\mathfrak{F} \cong \mathfrak{Fr}_\beta K$. Let $\mathfrak{A}\in K$ and let $k : G \twoheadrightarrow A$ be arbitrary. By [Ma78]Thm(19),p.150, there is a $\mathfrak{L}\in CA_3$ such that $\mathfrak{Ra}\,\mathfrak{L}= \mathfrak{A}$. Then $\mathfrak{L}\in K'$ by definition and $k : G \rightarrow Nr_2\mathfrak{L}$. Therefore k can be extended to a homomorphism $k' : \mathfrak{Fr}_\beta^{(\Delta)}K' \rightarrow \mathfrak{L}$. Then $k' : \mathfrak{Ra}\,\mathfrak{F} \rightarrow \mathfrak{Ra}\,\mathfrak{L} = \mathfrak{A}$ also holds. We have seen that $\mathfrak{Zd}\,\mathfrak{F}$ is not atomic. Then $\mathfrak{Nr}_2\mathfrak{F}$ is not atomic either, hence $\mathfrak{Fr}_\beta K$ $(\cong \mathfrak{Ra}\,\mathfrak{F})$ is not atomic, either. Thm.2(b) has been proved.

Now we give a direct proof for the case $K\subseteq RA$. Let $K\subseteq RA$ be such that $\overline{Eq}K$ is r.e. and $\mathfrak{R}(U)\in K$ for some infinite U. Let $\lambda\in Fm_\omega^0$ be an inseparable formula such that $\lambda\wedge\pi$ has a model. Such a λ exists by Lemma 2.7. Let $\eta \overset{d}{=} (rf\,\lambda)\cdot\pi_{RA}$. Let $H : \mathfrak{G} \rightarrow \mathfrak{Fr}_\beta K$ be a homomorphism such that H maps the free generator of $\mathfrak{G}$ to one of the free generators, say g, of $\mathfrak{Fr}_\beta K$. We will show that $H\eta \neq 0$ and there is no atom in $\mathfrak{Fr}_\beta K$ below $H\eta$. First we show $H\eta\neq 0$. By the Löwenheim-Skolem theorems, $\lambda\wedge\pi$ has a model with universe U, say $\mathfrak{M} = \langle U,E\rangle \models \lambda\wedge\pi$. For any $\varphi\in Fm_\omega^2$

let $\varphi^{\mathfrak{M}} \overset{d}{=} \{\langle a,b\rangle \in {}^2U : \mathfrak{M} \models \varphi[a,b]\}$. By Lemma 2.3(i) we have $p_i(x,y)^{\mathfrak{M}} = kH(rp_i(x,y))$ for $i \in 2$ and $(f\lambda)^{\mathfrak{M}} = kH(rf\lambda)$. By $\mathfrak{M} \models \pi$ then $kH(\pi_{RA}) = U\times U$. By $\mathfrak{M} \models \lambda \wedge \pi$ and Lemma 2.2(i), $(f\lambda)^{\mathfrak{M}} = \lambda^{\mathfrak{M}} = U\times U$. Thus $kH\eta = U\times U$, therefore $H\eta \neq 0$.

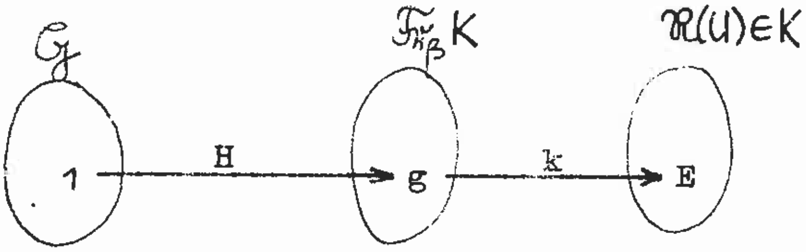


FIGURE 10

Assume now that $\gamma \leq H\eta$ is an atom in $\mathfrak{Fr}_\beta K$. Define $T \overset{d}{=} \{\varphi \in Fm^0_\omega : \gamma \leq Hrf\varphi\}$. Then T is decidable since γ is an atom and $\overline{Eq}K$ is r.e. (cf. the proof of Lemma 2.6). Assume that $\lambda \models \varphi$. Then $RRA \models (\pi_{RA} \cdot rf\lambda) \leq rf\varphi$ by Lemmas 2.2,3. Therefore $RA \models (\pi_{RA} \cdot rf\lambda) \leq rf\varphi$ by Tarski's representation theorem $QRA \subseteq RRA$. Hence $\gamma \leq rf\varphi$ by $\gamma \leq \pi_{RA} \cdot rf\lambda$. Thus $\varphi \in T$. Assume that $\lambda \models \neg\varphi$. Then $\gamma \leq rf(\neg\varphi) = -rf\varphi$ as before, hence $\gamma \not\leq rf\varphi$ since $\gamma \neq 0$. Thus $\varphi \notin T$. The above contradict the choice of λ, hence there is no atom in $\mathfrak{Fr}_\beta K$ below $H\eta$. QED(Thm.2(b))

Theorem 2(c) is proved in [N85c]. QED(Theorem 2)

§3. LOGICAL ASPECTS, OUTLINE OF A PURELY LOGICAL PROOF, ANSWERS TO A PROBLEM OF TARSKI, CONNECTIONS WITH SEMI-ASSOCIATIVE RELATION ALGEBRAS OF MADDUX

In this section, let our language Λ be $\Lambda=\langle\alpha,\underline{R},\varrho\rangle$ where $3\leq\alpha<\omega$, $\beta\overset{d}{=}Do\underline{R}$, $\mathfrak{R}\overset{d}{=}\{\underline{R}(i):i\in\beta\}$ and $(\forall i\in\beta)\ \varrho_i=2$. Thus the only difference is that we allow arbitrarily many binary relation symbols. We recall from §2 the following:
RAT denotes the set of relation algebraic terms (written up from the elements of $\mathfrak{R}$). If M is a set then $\mathcal{R}(M)$ denotes the relation algebra of all binary relations on M. The algebra $\mathfrak{Ora}$ was defined in Def.2.9 such that $Ora\subseteq Fm_3^1$ and $\mathfrak{Ora}$ is an algebra similar to RA's. We also recall from §2 that f, r, h are recursive functions for which the following hold:

$f : Fm_\omega^2 \longrightarrow Fm_3^2$ (Lemma 2.2)

$r : Fm_3^2 \longrightarrow RAT$ (Lemma 2.3)

$h : RAT \longrightarrow Fm_3^1$ (proof of Thm.1(a))

f,r, and h preserve "meaning" and Boolean structure", i.e.

(1) $\pi\models\varphi\leftrightarrow f\varphi$ for all $\varphi\in Fm_\omega^2$

(2) $\mathfrak{M}\models\varphi[a,b]\Longleftrightarrow\langle a,b\rangle\in m^{\mathfrak{M}}(r\varphi)$ for all $\varphi\in Fm_3^2$ and model $\mathfrak{M}=\langle M,R^{\mathfrak{M}}\rangle_{R\in\mathfrak{R}}$, where $m^{\mathfrak{M}}: RAT\to\mathcal{R}(M)$ is a homomorphism such that $m^{\mathfrak{M}}(R)\overset{d}{=}R^{\mathfrak{M}}$ for all $R\in\mathfrak{R}$,

(3) $h(R)\overset{d}{=}R(x_0x_1)\overset{d}{=}\exists yz(z=x_0\wedge y=x_1\wedge R(z,y))$ for all $R\in\mathfrak{R}$, $h:\mathfrak{Fr}_\beta\,SimRA\longrightarrow\mathfrak{Ora}$ is a homomorphism, and

$f(R(x,y)) = R(x,y)$, $f(\neg\varphi)=\neg f\varphi$, $f(\varphi\vee\psi)=f\varphi\vee f\psi$, $f(\varphi\wedge\psi)=f\varphi\wedge f\psi$, $r(R(x,y))=R$, $r(\neg\varphi)=-r\varphi$, $r(\varphi\vee\psi)=r\varphi+r\psi$, $r(\varphi\wedge\psi)=r\varphi\cdot r\psi$, $h(R)=R(x_0x_1)$, $h(-\tau)=pair(x)\wedge\neg h(\tau)$, $h(\tau+\sigma)=h\tau\vee h\sigma$, $h(\tau\cdot\sigma)=h\tau\wedge h\sigma$. Cf. Remark 2.4 in §2.

<u>DEFINITION 3.1.</u> We define $\mathcal{K} : Fm^2_\omega \longrightarrow Fm^1_3$ as follows: $\mathcal{K}\varphi \overset{d}{=} \forall x(pair(x) \to \mathcal{K}'\varphi)$ where $\mathcal{K}'\varphi \overset{d}{=} hrf(\varphi)$. $\mathcal{K} \overset{d}{=} \{\mathcal{K}\varphi : \varphi \in Fm^0_3\}$. □

<u>LEMMA 3.2.</u> Let $\varphi, \psi \in Fm^0_\omega$.

(i) $\vdash_3 \mathcal{K}(\varphi\wedge\psi) \leftrightarrow (\mathcal{K}\varphi \wedge \mathcal{K}\psi)$ and the same for $\vee$.

(ii) $\vdash_3 \mathcal{K}(\varphi \to \psi) \to (\mathcal{K}\varphi \to \mathcal{K}\psi)$

(iii) $\exists x pair(x) \vdash_3 \mathcal{K}(\neg\varphi) \to \neg\mathcal{K}\varphi$

(iv) $\mathcal{K}$ is decidable.

We omit the simple proof of Lemma 3.2. □

Recall that $\pi, \pi' \in Fm^0_3$, $\pi_{RA} \in RAT$ and $Ax \subseteq Fm^0_3$ were defined in §2 such that $\pi' \models Ax$ and Ax is finite. Define $\pi'_{RA} \overset{d}{=} \pi_{RA}\cdot([p;p^{\smile} \cdot q;q^{\smile}] \to 1')$ (where $p=r(p_0(x,y))$, $q=r(p_1(x,y)))$ and assume that $\pi\wedge h'(\pi'_{RA}) \in Ax$ where $h'(\pi'_{RA}) \overset{d}{=} \forall(pair(x) \leftrightarrow h(\pi'_{RA}))$. We note that π', π'_{RA} express that p,q form a pairing function such that pairs are unique; and $\pi' \models h'(\pi'_{RA})$ (hint: see the proof of Prop.3.3(i)-(ii)).

<u>Notation:</u> Let $T \subseteq Fm^0_\omega$, $\varphi \in Fm^0_\omega$. Then $T \models_{Ax} \varphi \overset{df}{\Longleftrightarrow} T \cup Ax \models \varphi$ and $T \vdash_{Ax} \varphi \overset{df}{\Longleftrightarrow} T \cup Ax \vdash_3 \varphi$.

PROPOSITION 3.3. $\varkappa: Fm^2_\omega \longrightarrow Fm_3$ is a recursive function for which (i)-(iv) below hold.

(i) $\pi \models \varphi \leftrightarrow \varkappa\varphi$ for all $\varphi \in Fm^0_\omega$.

(ii) $Ax \models \varphi$ iff $Ax \vdash_3 \varkappa\varphi$ for all $\varphi \in Fm^2_\omega$.

(iii) $T \models_{Ax} \varphi$ iff $\varkappa^* T \vdash_{Ax} \varkappa\varphi$ for all $T \subseteq Fm^0_\omega \ni \varphi$.

(iv) $T \models_{Ax} \varphi$ iff $T \vdash_{Ax} \varphi$ for all $T \subseteq \mathcal{K} \ni \varphi$.

Proof. Let $\varphi \in Fm^2_\omega$ and $\mathfrak{M}$ be a model of Λ, $\mathfrak{M} \models \pi$. Let $p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}$ be the "pairing functions" in $\mathfrak{M}$, for all $a \in M$ let $a_0 \overset{d}{=} p_0^{\mathfrak{M}}(a)$ and $a_1 \overset{d}{=} p_1^{\mathfrak{M}}(a)$ if they exist, and let $Pair^{\mathfrak{M}} \overset{d}{=} \{a \in M : a_0, a_1 \text{ exist}\}$. First we show that

(*) $\mathfrak{M} \models \varphi[a_0, a_1]$ iff $\mathfrak{M} \models \varkappa'\varphi[a]$, for all $a \in Pair^{\mathfrak{M}}$.

Let $\varphi \in Fm^2_\omega$ be arbitrary. Then $\mathfrak{M} \models \varphi[a_0, a_1]$ iff (by $\mathfrak{M} \models \pi$, $\pi \models \varphi \leftrightarrow f\varphi$) $\mathfrak{M} \models f\varphi[a_0, a_1]$ iff (by the properties of r) $\langle a_0, a_1\rangle \in m^{\mathfrak{M}}(rf\varphi)$. Therefore it is enough to show that

(**) $\langle a_0, a_1\rangle \in m^{\mathfrak{M}}(\tau)$ iff $\mathfrak{M} \models h\tau[a]$ for all $\tau \in RAT$.

We prove (**) by induction. Let $R \in \mathfrak{X}$. Then $\langle a_0, a_1\rangle \in m^{\mathfrak{M}}(R) = R^{\mathfrak{M}}$ iff $\mathfrak{M} \models R(x,y)[a_0, a_1]$ iff $\mathfrak{M} \models R(x_0 x_1)[a]$ iff $\mathfrak{M} \models h(R)[a]$. Assume that (**) holds for τ and σ. Then $\langle a_0, a_1\rangle \in m^{\mathfrak{M}}(-\tau)$ iff $\mathfrak{M} \models h(-\tau)[F]$ and $\langle a_0, a_1\rangle \in m^{\mathfrak{M}}(\tau + \sigma)$ iff $\mathfrak{M} \models h(\tau + \sigma)[a]$ are easy to see. We check now $\tau ; \sigma$. $\langle a_0, a_1\rangle \in m^{\mathfrak{M}}(\tau ; \sigma) = m^{\mathfrak{M}}(\tau) | m^{\mathfrak{M}}(\sigma)$ iff $(\exists b)(\langle a_0, b\rangle \in m^{\mathfrak{M}}(\tau) \wedge \langle b, a_1\rangle \in m^{\mathfrak{M}}(\sigma))$ iff (by $\mathfrak{M} \models \pi$, (**)) $(\exists c)(a_0 = c_{00} \wedge a_1 = c_{11} \wedge c_{01} = c_{10} \wedge \mathfrak{M} \models h\tau[c_0] \wedge \mathfrak{M} \models h\sigma[c_1])$ iff $\mathfrak{M} \models (h\tau^{\circ} h\sigma)[a]$ iff

$\mathfrak{M} \models h(\tau;\sigma)[a]$. The checking of $\tau^{\cup}$ is completely analogous, hence we omit it. (*) and (**) have been proved.

Proof of (i): Let $\varphi \in Fm^{0}_{\omega}$, let $a \in Pair^{\mathfrak{M}}$. Then $\mathfrak{M} \models \forall x(pair(x) \to \kappa'\varphi)$ iff $(\forall a \in Pair^{\mathfrak{M}})\ \mathfrak{M} \models \kappa'\varphi[a]$ if (by (*)) $(\forall a \in Pair^{\mathfrak{M}})\ \mathfrak{M} \models \varphi[a_0,a_1]$ iff (by $\varphi \in Fm^{0}_{\omega}$, $Pair^{\mathfrak{M}} \neq 0$) $\mathfrak{M} \models \varphi$.

Proof of (ii): $Ax \vdash_3 \kappa\varphi \Rightarrow Ax \models \varphi$ by soundness of $\vdash_3$ and by Prop.3.3(i). Next we prove $Ax \nvdash_3 \kappa\varphi \Rightarrow Ax \not\models \varphi$. Let $\mathfrak{R} \overset{d}{=} \mathfrak{Fm}/\equiv_{Ax}$. (Recall that $\equiv_{Ax}$ was defined in §2.) Then $\mathfrak{R} \in RA$ by Prop.2.10. Recall that $p_0(x,y), p_1(x,y) \in Fm^{2}_{3}$, $p = r(p_0(x,y))$ and $q = r(p_1(x,y))$. Let $\bar{p} \overset{d}{=} h(p)/\equiv_{Ax}$, $\bar{q} \overset{d}{=} h(q)/\equiv_{Ax}$. By $Ax \vdash_3 h'(\pi'_{RA})$ we have that

(1) $\mathfrak{R} \models \pi'_{RA}=1\ [p/\bar{p},\ q/\bar{q}]$,

(more precisely this means that $(\bar{p}^{\cup};\bar{p} \to 1')\cdot(\bar{q}^{\cup};\bar{q} \to 1')\cdot(\bar{p}^{\cup};\bar{q})\cdot([\bar{p};\bar{p}^{\cup}\cdot\bar{q};\bar{q}^{\cup}] \to 1')=1$ in $\mathfrak{R}$). Thus $\bar{p},\bar{q}$ are projection functions in $\mathfrak{R}$, hence $\mathfrak{R}$ is representable by Tarski's theorem (cf. [Ma78a]). Assume $Ax \nvdash_3 \kappa\varphi$. Then $Ax \nvdash_3 (\forall x)(pair(x) \to \kappa'\varphi)$, hence $Ax \nvdash_3 pair(x) \leftrightarrow \kappa'\varphi$, thus $\kappa'\varphi/\equiv_{Ax} \neq pair(x)/\equiv_{Ax} = 1^{\mathfrak{R}}$. Then by $\mathfrak{R} \in RRA$ there are a set U and a homomorphism $g : \mathfrak{R} \to \mathscr{R}(U)$ such that

(2) $g(\kappa'\varphi/\equiv_{Ax}) \neq 1^{\mathscr{R}(U)}$.

Let us fix such a g and U. Define $\mathfrak{M} \overset{d}{=} \langle U, R^{\mathfrak{M}}\rangle_{R\in\mathcal{R}}$ where $R^{\mathfrak{M}} \overset{d}{=} g(R(x_0x_1)/\equiv_{Ax})$ for each $R \in \mathcal{R}$. We will show that $\mathfrak{M} \models Ax$ while $\mathfrak{M} \not\models \varphi$. First we make an observation. For $\varphi \in Fm^{2}_{\omega}$ define $\varphi^{\mathfrak{M}} \overset{d}{=} \{\langle a,b\rangle : \mathfrak{M} \models \varphi[a,b]\}$. Recall that $m^{\mathfrak{M}} : \mathfrak{Fr}_{\mathcal{R}}SimRA \to \mathscr{R}(U)$ is a homomorphism such that $m^{\mathfrak{M}}(R)=R^{\mathfrak{M}}$ for every $R \in \mathcal{R}$ and $\varphi^{\mathfrak{M}}=m^{\mathfrak{M}}(r\varphi)$ for all $\varphi \in Fm^{2}_{3}$. Let $R \in \mathcal{R}$. Then $g(hR/\equiv_{Ax}) = g(R(x_0x_1)/\equiv_{Ax})=R^{\mathfrak{M}}=m^{\mathfrak{M}}(R)$. Since g,h are homomorphisms, this implies that

(*) $g(h(r\psi)/{\equiv_{Ax}}) = m^{\mathfrak{M}}(r\psi) = \psi^{\mathfrak{M}}$ for all $\psi \in Fm_3^2$ see Fig.11 below Now we are ready to prove $\mathfrak{M} \models Ax$ and $\mathfrak{M} \not\models \varphi$. By $p_0(x,y), p_1(x,y) \in Fm_3^2$ and (*) we have that $g(\bar{p}) = p_0(x,y)^{\mathfrak{M}}$, and $g(\bar{q}) = p_1(x,y)^{\mathfrak{M}}$, hence by (1) we have that $\mathfrak{M} \models \pi'$. Hence $\mathfrak{M} \models Ax$ by $\pi' \models Ax$. By (2) we have $g(h(rf\varphi)/{\equiv_{Ax}}) \neq U \times U$, hence $(f\varphi)^{\mathfrak{M}} \neq U \times U$ by (*), i.e. $\mathfrak{M} \not\models f\varphi$. Above we have seen that $\mathfrak{M} \models \pi$

$$\begin{tikzcd}
\mathfrak{Fr}_{\mathcal{R}}\mathrm{SimRA} \arrow[r, "h"] \arrow[rrr, bend right, "m^{\mathfrak{M}}"'] & \mathfrak{A} \arrow[r, "\equiv_{Ax}"] & \mathfrak{R} \arrow[r, "g"] & \mathcal{R}(U)
\end{tikzcd}$$

FIGURE 11

hence by Prop.3.3(i) we also have that $\mathfrak{M} \not\models \varphi$. (i) has been proved. Proof of (iii): $T \models_{Ax} \varphi$ implies $Ax \models \wedge T \rightarrow \varphi$ implies (by 3.3(ii)) $Ax \vdash_3 \kappa(\wedge T \rightarrow \varphi)$ implies (by L.3.2(i)-(ii)) $Ax \vdash_3 \wedge \kappa^* T \rightarrow \kappa\varphi$ implies (by the deduction theorem) $Ax \cup \kappa^* T \vdash_3 \kappa\varphi$ implies (by soundness of $\vdash_3$) $Ax \cup \kappa^* T \models \kappa\varphi$ implies (by $Ax \models \pi$, Prop.3.3(i)) $T \models_{Ax} \varphi$. Proof of (iv): Let $T \subseteq K \ni \varphi$. Then there are $\Sigma \subseteq Fm_\omega^0 \ni \psi$ such that $T = \kappa^* \Sigma$ and $\varphi = \kappa\psi$. Now $T \models_{Ax} \varphi$ means $\kappa^* \Sigma \models_{Ax} \kappa\psi$ implies (by $Ax \models \pi$, Prop.3.3(i)) $\Sigma \models_{Ax} \psi$ implies (by Prop.3.3(iii)) $\kappa^* \Sigma \vdash_{Ax} \kappa\psi$, i.e. $T \vdash_{Ax} \varphi$ implies (by soundness) $T \models_{Ax} \varphi$. (We note that we may assume $|T| < \omega$ by compactness.) QED

REMARK 3.4. One can prove Prop.3.3(ii) in a purely logical manner, without using Tarski's representation theorem $[\mathfrak{R} \in RA,\ \mathfrak{R} \models \pi_{RA} \Rightarrow \mathfrak{R} \in RRA]$, as follows. Recall that $\vdash_\omega$ is a complete proof system (see Remark 1.2(a)). Let $\mathfrak{Ax}_\omega$ denote the set of logical axiom schemes ⟪1⟫ - ⟪9⟫ of $\vdash_\omega$. One proves first that $Ax \vdash_3 \kappa\gamma$ for all logical axioms $\gamma \in \mathfrak{Ax}_\omega$, by the methods of the proof of Prop.2.10. Let now $\Gamma \overset{d}{=} \{\gamma \in Fm_\omega^0 : Ax \vdash_3 \kappa\gamma\}$. Then Γ contains all the logical axioms $\mathfrak{Ax}_\omega$ and is closed under the inference rules since $\vdash_3 \kappa(\varphi \rightarrow \psi) \rightarrow (\kappa\varphi \rightarrow \kappa\psi)$ by L.3.2. Therefore Γ contains all ω-derivable, hence all valid sentences i.e. $(\forall \varphi \in Fm_\omega^0)[\models \varphi \Rightarrow Ax \vdash_3 \kappa\varphi]$.

Next one proves that $Ax \vdash_3 \kappa(Ax)$ (again, by using the techniques of the proof of Prop.2.10). Now, $Ax \models \varphi \Rightarrow \models Ax \to \varphi \Rightarrow Ax \vdash_3 \kappa(Ax) \to \kappa\varphi \Rightarrow Ax \vdash_3 \kappa\varphi$. So we have $(\forall \varphi \in Fm^0_\omega)[Ax \models \varphi \Leftrightarrow Ax \vdash_3 \kappa\varphi]$. But this is a completeness theorem for $Rg\kappa$ and $\vdash_3$ w.r.t. simulating (via κ) full first-order logic. But then $\vdash_3$ inherits all properties of first-order logic (including Gödel's incompleteness). To finish the (second) purely logical proof of our main theorem, Thm.1(a), next we show how Prop.3.3(ii) implies that L_3 has G.i. Let $\lambda \in Fm^0_\omega$ be an inseparable sentence, i.e. such that there is no decidable set $T \subseteq Fm^0_\omega$ such that $(\forall \varphi \in Fm^0_\omega)[\lambda \models \varphi \Rightarrow \varphi \in T)$ and $(\lambda \models \neg\varphi \Rightarrow \varphi \notin T)]$. There exist such sentences, see e.g. [E72],[M76]. Let $\psi \overset{d}{=} Ax \wedge \kappa\lambda$, where Ax is defined in §2. We will show that $\psi \in Fm_3$ is not "completable". Assume that $T' \subseteq Fm^0_3$ is such that $\psi \in T'$, $T' = \{\varphi \in Fm^0_3 : T' \vdash_3 \varphi\}$, $(\forall \varphi \in Fm^0_\omega)(\varphi \in T'$ iff $\neg\varphi \in T')$ and that T' is decidable. Define $T \overset{d}{=} \{\varphi \in Fm^0_\omega : \kappa\varphi \in T'\}$. Clearly, T is decidable. Assume that $\lambda \models \varphi$. Then $\models \lambda \to \varphi$, hence $Ax \vdash_3 \kappa\lambda \to \kappa\varphi$ by Prop.3.3(ii) and the properties of κ, therefore $\kappa\varphi \in T'$ since $Ax, \kappa\lambda \in T'$ by $\psi \in T'$. Thus $\varphi \in T$. Assume that $\lambda \models \neg\varphi$. Then $\kappa(\neg\varphi) \in T'$. By L.3.2 we have $\pi \vdash_3 \kappa(\neg\varphi) \to \neg\kappa\varphi$, thus $\neg\kappa\varphi \in T'$, therefore $\kappa\varphi \notin T'$, i.e. $\varphi \notin T$. The above contradict the choice of λ, hence ψ is incompletable in Fm_3. G.i. for L_3 has been proved.

Tarski's original proof for $(\mathfrak{A} \in RA,\ \mathfrak{A} \models \pi_{RA} \Rightarrow \mathfrak{A} \in RRA)$ was similar to the above chain of thoughts, see [T53a],[TG]. However, today there exists a simple, purely algebraic proof for Tarski's above mentioned representation theorem, see

Maddux[Ma78a]. The question arises: if Tarski proved his representation theorem from the analogous logical theorem, cannot one prove then an analogous representation theorem for the wider class SA of algebras? (The reason to think so is that while RA is the class naturally corresponding to $\vdash_4$-provability, SA is the class naturally corresponding to $\vdash_3$, see [Ma78,82,83].) We shall return to this question later, in Remarks 3.12,13. $\square$

In the next proposition we discuss Prop.3.3. We will show that in Prop.3.3(i) one cannot replace $\models$ with $\vdash_3$ and in Prop.3.3(ii) one cannot replace Ax with all $T \supseteq Ax$.

<u>PROPOSITION 3.5.</u> (i) $Ax \nvdash_3 (\varphi \to \varkappa\varphi)$ for some $\varphi \in Fm_3^0$.

(ii) $Ax \nvdash_3 (\varkappa\varphi \to \varphi)$ for some $\varphi \in Fm_3^0$.

(iii) $T \models \varphi \;\not\Rightarrow\; T \vdash_3 \varkappa\varphi$ for some $Ax \subseteq T \subseteq Fm_3^0$, $\varphi \in Fm_3^0$.

(iv) $T \models \varkappa\varphi \;\not\Rightarrow\; T \vdash_3 \varphi$ for some $Ax \subseteq T \subseteq Fm_3^0$, $\varphi \in Fm_3^0$.

<u>Proof.</u> Recall the notations Q^+CA_3 and $Fn\mathfrak{A}$ from Def.3.8 and from the beginning of the proof of Thm.3.7 respectively. By the methods of the proof of Thm.3.7, one can prove the following (the proof can be found in the appendix):

(*) There is a $\mathfrak{L} \in Q^+CA_3$ generated by a single element of $Nr_2\mathfrak{L}$ such that $(\exists f,g \in Fn\mathfrak{L})f;g \notin Fn\mathfrak{L}$.

Let $\mathfrak{L} \in Q^+CA_3$, $f,g \in Fn\mathfrak{L}$ be as in (*) and let $C=Sg\{e\}$ for $e \in Nr_2\mathfrak{L}$. Let $t : \mathfrak{Fm}_3 \to \mathfrak{L}$ be the homomorphism for which $t(E(x,y)) \overset{d}{=} e$. Then t is onto $\mathcal{L}$. Therefore there are $\varphi, \gamma \in Fm_3^2$ such that $t(\varphi)=f$ and $t(\gamma)=g$. Let us fix such a φ and γ. Let ψ be the formula $(\in Fm_3^0)$ expressing that " φ and γ are functions $\rightarrow$ $\varphi;\gamma$ is a function". More precisely: Let $^{\cup}$, $;$ denote the operations of $\mathfrak{Ra}\,\mathfrak{Fm}_3$ (cf. [HMT]5.3.7), i.e. $\varphi^{\cup} = s_1^0 s_2^1 s_0^2 \exists v_2 \varphi$ and $\varphi;\psi \overset{d}{=} \exists v_2(s_2^1\varphi \wedge s_2^0\psi)$ (recall the notation $s_j^i\varphi$ from the beginning of the proof of L.2.3.). Now ψ is the following formula:

$$\forall xy[(\varphi^{\cup};\varphi \rightarrow x=y) \wedge (\gamma^{\cup};\gamma \rightarrow x=y)] \rightarrow \forall xy[(\varphi;\gamma)^{\cup};(\varphi;\gamma) \rightarrow x=y] .$$

Then $t(\psi)=0^{\mathcal{L}}$ by $f,g \in Fn\mathcal{L}$, $f;g \notin Fn\mathcal{L}$. Define $T \overset{d}{=} \{ \delta \in Fm_3^0 : t(\delta)=1^{\mathcal{L}} \}$. Then $\neg\psi \in T$, $\psi \notin T$ by the above. Also $Ax \subseteq T$ and $[T \vdash_3 \varrho \Rightarrow \varrho \in T]$ i.e. $\hat{T} = T$ by $\mathcal{L} \in Q^+CA_3$, hence

(1) $Ax \vdash_3 \varrho \quad \Rightarrow \quad \varrho \in T$ for every $\varrho \in Fm_3^0$.

Clearly, $Ax \models \psi$. Hence $Ax \vdash_3 \varkappa\psi$ by Prop.3.3, thus $\varkappa\psi \in T$ and $\neg\varkappa\psi \notin T$ by (1). By $\psi \notin T$, $Ax \vdash_3 \varkappa\psi$ and (1) we have

(2) $Ax \nvdash_3 \varkappa\psi \rightarrow \psi$

hence (ii) has been proved. By Lemma 3.2 (and $Ax \vdash_3 \exists x pair(x)$) we have $Ax \vdash_3 \varkappa(\neg\psi) \rightarrow \neg\varkappa\psi$. Hence by $\neg\varkappa\psi \notin T$ we have $\varkappa(\neg\psi) \notin T$. By $\neg\psi \in T$ and (1) we then get

(3) $Ax \nvdash_3 \neg\psi \rightarrow \varkappa(\neg\psi)$.

(i) has been proved. Now $\hat{T} = T$, $\neg\psi \in T$, $\varkappa(\neg\psi) \notin T$, and $\varkappa\psi \in T$, $\psi \notin T$ prove (iii) and (iv). QED

We now show that Prop.3.3 immediately yields one of the main decidability results of CA theory, due to Maddux [Ma80]. His theorem, Corollary 3.6(a) below ($\overline{Eq}CA_3$ is undecidable), solves a problem of Tarski which was open till 1977, see [HMT] Part II p.(vi). We note that our proof of Cor.3.6(a) is completely analogous to Tarski's original proof of "$\overline{Eq}$RA is undecidable" which used statements analogous to Prop.3.3(ii). (Cf. [TG]p.0.10, Tarski's theorem was announced in [T41] and the proof was outlined in Lemmas I-III of [T53].)

Cor.3.6(b) is a slight generalization of Maddux's result. (His proof can be used to obtain a similar result with $\mathfrak{M}$ replaced by the free semigroup with infinitely many free generators.) By passing we note that in [N85b] a stronger undecidability result is available (in this direction) which is proved by a generalization of the methods herein, but which can also be proved by generalizing Maddux's original method.

To state the following, we recall that Bo_α and the axioms $(C_0 - C_7)$ defining CA_α are found in [HMT], and to any model $\mathfrak{M}$, a cylindric algebra $\mathfrak{Cs}^{\mathfrak{M}} \in CA_\alpha$ is associated in [HMT]§4.3 (cf. also §1.5).

COROLLARY 3.6. ([Ma80]) (a) $\overline{Eq}CA_3$ is undecidable.

(b) Let $3 \leq \alpha < \omega$. Let $K \subseteq Bo_\alpha$ satisfy $(C_2-C_4),(C_7)$. Let $\mathfrak{M}$ be any model of Peano's arithmetic (PA). Assume $\mathfrak{Cs}^{\mathfrak{M}} \in K$. Then $\overline{Eq}K$ is undecidable.

Proof. Before turning to the proof, we note that PA could be replaced in 3.6(b) with the weaker (and finite) $A_E + \pi'$ where π' is written up for the natural (or usual) arithmetical pairing functions p_0, p_1. Actually, PA could be replaced with any inseparable theory $+ \pi'$. (a) is clearly a corollary of (b). First we prove (b) with the additional hypothesis that $K \subseteq CA_\alpha$ (and later we shall eliminate this hypothesis). Recall from [HMT]§4.3 that there is $\tau\mu : Fm^0_3 \rightarrow$ "CA_3-terms" such that $\varphi \vdash_3 \psi$ iff $CA_3 \models \tau\mu(\varphi) \leq \tau\mu(\psi)$ for every $\varphi,\psi \in Fm^0_3$. Let λ be an inseparable formula such that $\mathfrak{M} \models \lambda \wedge \pi'$. Then $\mathfrak{M} \models Ax \wedge \varkappa\lambda$, too. Define $T \overset{d}{=} \{\varphi \in Fm^0_\omega : K \models \tau\mu(Ax \wedge \varkappa\lambda) \leq \tau\mu(\varkappa\varphi)\}$. Let $\varphi \in Fm^0_\omega$. Assume that $\lambda \models \varphi$. Then $Ax \wedge \lambda \models \varphi$, hence $Ax \wedge \varkappa\lambda \vdash_3 \varkappa\varphi$ by Prop.3.3(iv). Therefore $K \models \tau\mu(Ax \wedge \varkappa\lambda) \leq \tau\mu(\varkappa\varphi)$, i.e. $\varphi \in T$. Assume now $\lambda \models \neg\varphi$. Then $Ax \wedge \varkappa\lambda \vdash_3 \neg\varkappa\varphi$ by the above and by L.3.2, hence $K \models \tau\mu(Ax \wedge \varkappa\lambda) \leq -\tau\mu(\varkappa\varphi)$. By $\mathfrak{M} \models Ax \wedge \varkappa\lambda$ and $\mathfrak{Cs}^{\mathfrak{M}} \in K$ we have $K \not\models \tau\mu(Ax \wedge \varkappa\lambda) = 0$, hence $K \not\models \tau\mu(Ax \wedge \varkappa\lambda) \leq \tau\mu(\varkappa\varphi)$ by the above, i.e. $\varphi \notin T$. We have seen that T separates the theorems of λ from the refutable sentences of λ, hence T is not recursive. Since $\tau\mu$ and $\varkappa$ are recursive, this implies that $\overline{\mathrm{Eq}}K$ is not recursive.

Assume now the hypotheses of Cor.3.6(b) (without assuming $K \subseteq CA_\alpha$). Let $\sigma \overset{d}{=} \sum\{(c_i 0 \oplus 0)+(d_{ii} \oplus 1)+(d_{ik} \oplus c_j(d_{ij} \cdot d_{jk}))$: $i,j,k \in \alpha$, $j \notin \{i,k\}\}$, where $\oplus$ denotes symmetric difference. Then $(\forall \mathfrak{A} \in Bo_\alpha)[\mathfrak{A} \models \sigma=0$ iff $\mathfrak{A} \models C_1, C_5, C_6]$. Let $\delta \overset{d}{=} -c_{(\alpha)}\sigma$. Then $rl_\delta \in Ho\mathfrak{A}$ for every $\mathfrak{A} \in K$ by [N80]Thm.1(i) since $\mathfrak{A} \models c_i c_{(\alpha)} \sigma = c_{(\alpha)} \sigma$ for every $i \in \alpha$ by $\mathfrak{A} \models C_3, C_4$. Let $\mathfrak{A} \in K$, $\mathfrak{B} \overset{d}{=} rl_\delta^{*}\mathfrak{A}$. Then $\mathfrak{B} \in CA_\alpha$ since $\mathfrak{B} \models C_1, C_5, C_6$ by $\mathfrak{B} \models \delta=1$. Also, $\mathfrak{B} \models \tau=\varrho$ iff $\mathfrak{A} \models \delta \cdot \tau = \delta \cdot \varrho$ for any terms τ, ϱ. Let $K' \overset{d}{=} \{rl_\delta^{*}\mathfrak{A} : \mathfrak{A} \in K\}$. Then $K' \subseteq CA_\alpha$ and ($\overline{\mathrm{Eq}}K$ decidable $\Rightarrow$ $\overline{\mathrm{Eq}}K'$ decidable). By $\mathfrak{L} \overset{d}{=} \mathfrak{Ls}^{\mathfrak{M}} \in CA_\alpha$ we have that $rl_\delta^{*}\mathfrak{L} = \mathfrak{L}$, hence $\mathfrak{L} \in K'$. Therefore $\overline{\mathrm{Eq}}K'$ is undecidable by our previous proof. Thus $\overline{\mathrm{Eq}}K$ is undecidable, too.
<u>QED</u>

The following theorem states that representability of QRA does not carry over to "QSA" and "QCA_3" no matter how we define the latter two classes (i.e. not even if we strengthen the definition of pairing function elements p,q by adding further postulates on p and q e.g. like in $\mathfrak{N}'$). Recall that if $\mathfrak{A} \in SimRA$ then $Fn\mathfrak{A} \overset{d}{=} \{a \in A : a^\smile ; a \leq 1'\}$.

<u>THEOREM 3.7.</u> There are $\mathfrak{A} \in SA \sim RA$ and $p,q \in Fn\mathfrak{A}$ with $p^\smile ; q = 1$ such that (i)-(ii) below hold.

(i) There is $\mathfrak{G} \subseteq \mathfrak{A}$ with $p,q \in Fn\mathfrak{G}$ such that $\mathfrak{G} \in QRA \subseteq RRA$. Moreover $\mathfrak{G}$ is a "<u>standard</u>" QRA in the following sense: $\mathfrak{G} \subseteq \mathfrak{R}(U)$ for some set U and $p = pj_0 \upharpoonright U$ and $q = pj_1 \upharpoonright U$ where pj_i is the standard set theoretic i-th projection function (i.e. $\langle a,b\rangle \xmapsto{pj_0} a$ and $\langle a,b\rangle \xmapsto{pj_1} b$). In other words $Doq = Dop = \{\langle a,b\rangle : a,b\in U\}$ and $(\forall a,b\in U)[p\langle a,b\rangle = a \;\&\; q\langle a,b\rangle = b]$.

(ii) $\mathfrak{A}\notin RRA$, moreover $(\exists f,h\in Fn\,\mathfrak{A})f;h\notin Fn\,\mathfrak{A}$, i.e. $\mathfrak{A}\not\models$ "composition of functions is a function".

Before proving Thm.3.7 we state some corollaries (and related results).

DEFINITION 3.8. (i) Let $K\subseteq SimRA$. Let $\mathfrak{A}\in K$. We define $\mathfrak{A}\in QK$ iff $(\exists p,q\in Fn\,\mathfrak{A})\; p^{\cup};q=1$ and $\mathfrak{A}\in Q^{+}K$ iff $(\exists p,q\in Fn\,\mathfrak{A})[p^{\cup};q=1$ and $(p;p^{\cup})\cdot(q;q^{\cup})\leq 1']$.

(ii) Let $\alpha\geq 3$ and $K\subseteq Bo_{\alpha}$. Then $Q^{+}K \overset{d}{=} \{\mathfrak{A}\in K : \mathfrak{Ra}\,\mathfrak{A}\in Q^{+}SimRA$ and $A=Sg^{\mathfrak{A}}Nr_2\mathfrak{A}\}$. QK is defined analogously.

(iii) A QSimRA $\mathfrak{A}$ with $p,q\in Fn\mathfrak{A}$ is *standard* if to $\mathfrak{G}\overset{d}{=}\mathfrak{Sg}^{\mathfrak{A}}\{p,q\}$ there is an isomorphism $h:\mathfrak{G}\rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{R}(H_{\omega})$ where $H_{\omega}=\bigcup\{H_i : i\in\omega\}$ with $(\forall n\in\omega)H_{n+1}=H_n\cup{}^{2}(H_n)$ such that $h(p)$ and $h(q)$ are the standard set theoretic projection functions.

(iv) $\mathfrak{A}\in QCA_{\alpha}$ is standard if $\mathfrak{Ra}\,\mathfrak{A}$ is such. □

Note that any standard QK is actually a $Q^{+}K$. Further, QK corresponds to our formula π and $Q^{+}K$ corresponds to π'.

COROLLARY 3.9. (i) The elements of QSA and QCA_3 are not representable in general.

(ii) There is $\mathfrak{A}\in QCA_3$ such that $\mathfrak{Ra}\,\mathfrak{A}$ is not representable.

(iii) Statements (i) and (ii) above remain true if we strengthen the definition of QK the same way as it was done in Thm.3.7(i).

PROPOSITION 3.10. There is $\mathfrak{A}\in QCA_3$ such that $\mathfrak{Ra}\,\mathfrak{A}\notin SA$. Again a strengthened version like Thm.3.7(i) holds, too.

We shall return to the proof later. In passing we note that there is $\mathfrak{A}\in Q^+SA$ with projections p,q such that $p;p\notin Fn\,\mathfrak{A}$.

Proof of Thm.3.7: Notation(RA-terms): $dom(x)\overset{d}{=}(x;1)\cdot 1'$, $rng(x)\overset{d}{=}(1;x)\cdot 1'$ and $0'\overset{d}{=}-1'$.

Let $\mathfrak{A}\in QRA$ and $p,q\in Fn\,\mathfrak{A}$ be arbitrary but satisfying (I)-(III) below:

(I) $p^{\cup};q=1$ and $dom(p)=dom(q)$

(II) there is $e\in At\,\mathfrak{A}$ with $e\leq(1'-dom(p))$

(III) $(\exists f,g,h\in At\,\mathfrak{A}\cap Fn\,\mathfrak{A})dom(f)=dom(g)=dom(h)=rng(f)=rng(g)$
$=rng(h)=e$ and $g^{\cup}=g$, $f^{\cup}=f$, $g;h=f$.

We do not really need $g^{\cup}=g$ and $f^{\cup}=f$, these only serve to make some computations shorter. Let $G\overset{d}{=}\{e,g,f,h\}$. We shall construct a nonrepresentable QSA out of $\mathfrak{A}$.

Remark: We note that condition (III) is not essential in our method, we stated it only to have a few simple elements to work with. Also $e\in At\,\mathfrak{A}$ is not essential. We could start out with almost any structure S below e (more precisely, below $(e;1)\cdot(1;e)$) instead of G and then construct a nonrepresentable version of S while leaving the rest of $\mathfrak{A}$ unchanged.

It is easy to see that such an $\mathfrak{A}$ exists. For example, Fig. 12 below illustrates such a construction, where of course p and q are the standard set theoretic projection functions. Further f, g, and h are three functions on H_0, $e = Id \restriction H_0$ and the base of $\mathfrak{A}$ is H_ω.

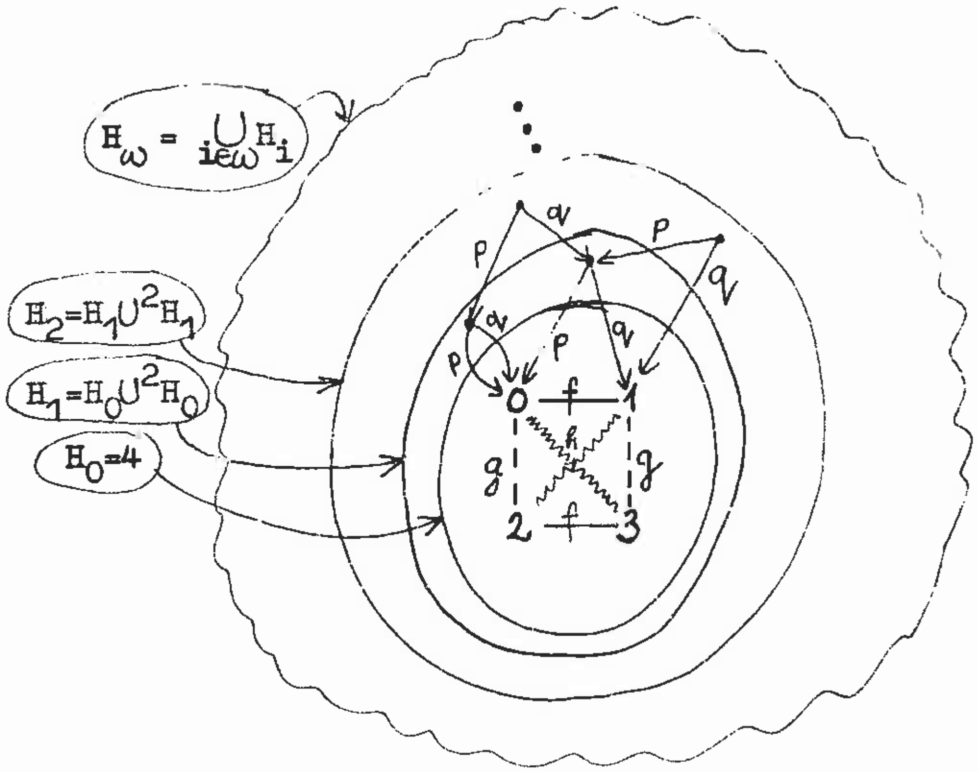
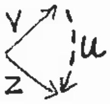


FIGURE 12

Throughout the rest of the proof, we shall heavily use the notation of [Ma82] concerning complex algebras and atom structures for SA's. Recall from §2 therein that an SA atom-structure is a structure $\mathfrak{A} = \langle U, C, k, I\rangle$ where $C \subseteq {}^3U$, $k \in {}^UU$ and $I \subseteq U$. If $\mathfrak{A}$ is an SA-atom-structure then $\mathfrak{Cm}\,\mathfrak{A}$ is similar to SA's as defined in Def.2.1 in [Ma82]. On the other hand, if $\mathfrak{A}$ is a Bo_α (i.e. CA-like) atom structure then $\mathfrak{Cm}\,\mathfrak{A}$ is a Bo_α as defined in [HMT]. (These two definitions do not contradict each other.) Thm.2.2(4) of [Ma82] says that $\mathfrak{Cm}\,\mathfrak{U} \in SA$ iff $\mathfrak{U}$ satisfies conditions (a-c),(e) formulated therein. Therefore $\mathfrak{A} \cong|\subseteq \mathfrak{Cm}\,\mathfrak{U} \in RA$ for some $\mathfrak{U} = \langle U,C,k,I\rangle$ by Thm.4.3(4) of [Ma82] since $\mathfrak{A} \in RA$. For simplicity, we assume $\mathfrak{A} \subseteq \mathfrak{Cm}\,\mathfrak{U}$ and $At\,\mathfrak{A} \subseteq At\,\mathfrak{Cm}\,\mathfrak{U}$. The cycle $[f,g,f]=\{\langle f,g,f\rangle,\langle g,f,f\rangle,\langle f,f,g\rangle\}$ was defined in [Ma82], [Ma84]. We define $C^+ \overset{d}{=} C \cup [f,g,f]$. Let $\mathfrak{U}^+ \overset{d}{=} \langle U,C^+,k,I\rangle$. Clearly $\mathfrak{U}^+$ satisfies (a)-(c) of the quoted Thm.2.2(4). To check (e), let $\langle v,w,x\rangle,\langle x,y,z\rangle \in C^+$, in figure 

.

To satisfy (e), we need an $u \in U$ with $\langle v,u,z\rangle \in C^+$ that is we need a cycle 

. If $v,z \in G$ then there is such a "u" by the original definition of $\mathfrak{U}$. ($e \neq v \neq z \neq e \Rightarrow u$ is the remaining element of G, and $e=v \Rightarrow u=z$, $v=z \Rightarrow u=e$, and $z=e \Rightarrow u=v$). Therefore we may assume $v \in G \Rightarrow z \notin G$. If $\langle v,w,x\rangle$, $\langle x,y,z\rangle \in C$ then we are done (since C did satisfy (e) originally. Hence we may assume $\langle v,w,x\rangle \in [f,g,f]$ or $\langle x,y,z\rangle \in [f,g,f]$. Thus one of cases 1-6 below holds.

<u>Case 1</u> $\langle v,w,x\rangle=\langle f,g,f\rangle$. 

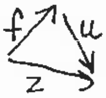
Then $u=y$ will do since $\langle f,y,z\rangle \in C$ already (since $z \notin G$ hence $\langle f,y,z\rangle \in C$).

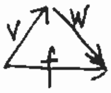
<u>Case 2</u> $\langle v,w,x\rangle=\langle g,f,f\rangle$, i.e. 

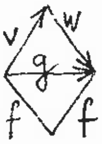
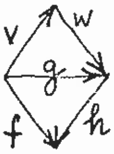
in C^+. By our assumption $v \in G \Rightarrow z \notin G$, made before this case distinction, we have $z \notin G$. Warning: we shall not repeat this argument in the remaining cases. Thus 

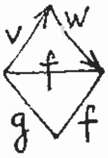
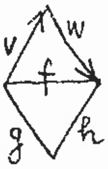
is in C which satisfies (e), hence $(\exists\, u \in U)$ 

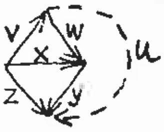
is in C. This u will do.

<u>Case 3</u> $\langle v,w,x\rangle=\langle f,f,g\rangle$, i.e.  in C^+. Then  in C. Thus  for some u in C.

<u>Case 4</u> $\langle x,y,z\rangle=\langle f,g,f\rangle$, i.e.  in C^+. Then $v\notin G$, hence  in C, hence u=w will do.

<u>Case 5</u> $\langle x,y,z\rangle=\langle g,f,f\rangle$, i.e.  . Then  in C, hence  for some u.

<u>Case 6</u> $\langle x,y,z\rangle=\langle f,f,g\rangle$, i.e.  . Then  in C, hence  for some u.

Cases 1–6 above show that there is indeed a $u\in U$ with $\langle v,u,z\rangle \in C^+$ that is filling in the diagram  . Actually, we proved more, namely $\langle v,u,z\rangle\in C$ already, that is $v;u\geqslant z$ holds in $\mathfrak{A}$. This proves that $\mathfrak{A}^+$ satisfies (a)–(c),(e). Therefore $\mathfrak{B}\overset{d}{=}\mathfrak{Em}\,\mathfrak{A}^+\in SA$ by Thm.2.2 of [Ma82]. Since $A\subseteq B$ we can ask how p,q behave in $\mathfrak{B}$. Let $t=(e;1)\cdot(1;e)$. Since $(\forall x\in G)x\leqslant t$ and p,q and $p^{\cup}$ and $q^{\cup}$ are disjoint from t, $;^{\mathfrak{A}}$ and $;^{\mathfrak{B}}$ agree on $\{p,q,p^{\cup},q^{\cup}\}$ (since the only new cycle added to C was [f,g,f] which is below t). Therefore we proved

(*) $p,q\in Fn\mathfrak{B}$ and $p^{\cup};q=1$ in $\mathfrak{B}$. Hence $\mathfrak{B}\in QSA$.

On the other hand, $\mathfrak{B}$ is not representable (see Claim 2 below), so we proved part of what we wanted. We want to prove more:

We want to prove that $\langle \mathfrak{Sg}^{\mathfrak{B}}\{p,q\}, p,q\rangle$ is "standard" (or has a standard representation). But for this we need more assumptions on $\mathfrak{A}$ since

(**) $(p;p^{\cup})\cdot(q;q^{\cup}) \leq 1'$ and $(p+q)\cdot 1'=0$ and $p^n\cdot 1'=0$ (for $n\in\omega$)

do hold in standard QRA's but do not hold in arbitrary QRA's. So let us assume that p,q of $\mathfrak{A}$ is a standard pair of projections, that is assume

(***) $\mathfrak{A}\subseteq\mathfrak{R}(W)$ and $W=\bigcup\{H_i : i\in\omega\}$ and $(\forall n\in\omega)H_{n+1}=H_n\cup{}^2H_n$ and H_0 is an arbitrary set. Further, $p,q : W\to W$ are the set theoretic projections.

Such an $\mathfrak{A}$ satisfying (I)-(III) in addition to (***) obviously exists, for example the algebra constructed below the remark following (I)-(III) is such. Let W and H_n as in (***) be fixed for the rest of the proof. Let $Q \overset{d}{=} Sg^{\mathfrak{A}}\{p,q\}$. Let $T = (H_0\times H_0)\sim Id$. Then $T\in Q$ since $T = \tau(1'-(p;1))$ where $\tau(x) = (1;x)\cdot(x;1) - 1'$.

<u>Claim 1</u> T is an atom of Q .

<u>Proof.</u> Let $k : H_0 \twoheadrightarrow H_0$ be an arbitrary permutation of H_0. Then there is a permutation $f : W \twoheadrightarrow W$ of W with $f\supseteq k$ such that $\tilde{f}(p)=p$ and $\tilde{f}(q)=q$ where $\tilde{f}R \overset{d}{=}\{\langle fa,fb\rangle : \langle a,b\rangle\in R\}$. Thus the base-automorphism $\tilde{f}\in Is(\mathfrak{A},\mathfrak{A})$ is such that $Q\upharpoonright\tilde{f}\subseteq Id$. Thus

(+) $(\forall x\in Q)\left[\tilde{f}x=x \text{ hence } \tilde{k}x=x\right]$.

Let $\langle a,b\rangle,\langle c,d\rangle\in T$. Then there is a permutation $k : H_0 \twoheadrightarrow H_0$

with $\langle ka,kb\rangle=\langle c,d\rangle$ (since $a\neq b$ and $c\neq d$ by $T\leq 0'$). Thus $(\forall x\in Q)\left[\langle a,b\rangle\in x \iff \langle c,d\rangle\in x\right]$. This means that T is an atom of Q. QED(Claim 1)

Thus $p,q,T\in Q\subseteq Su\mathfrak{A}$ and T is an atom of Q. Clearly $Q\subseteq A\subseteq B = SbU$. Is $Q\in Su\mathfrak{B}$, too? The Boolean operations, $\cup$ and $1'$ are the same in $\mathfrak{A}$ and $\mathfrak{B}$. The only operation that is different is composition (;). Let $x,y\in A$. Then $x\cup y\subseteq U$ and $x;^{\mathfrak{A}}y = \{c : \langle a,b,c\rangle\in C$ and $a\in x,\ b\in y\}$ while $x;^{\mathfrak{B}}y = \{c : \langle a,b,c\rangle\in C^+$ and $a\in x,\ b\in y\}$. Assume $x;^{\mathfrak{A}}y\neq x;^{\mathfrak{B}}y$. Then there is $\langle a,b,c\rangle\in C^+\sim C$ with $a\in x$, $b\in y$ and $c\in(x;^{\mathfrak{B}}y)\sim x;^{\mathfrak{A}}y$. By the definition of C^+ then $\langle a,b,c\rangle\in[f,g,f]$. Then f or g is in x and f or g is in y. But since $f\in T\in AtQ$ and $x,y\in Q$, we have $T\leq x$ and $T\leq y$. But then $f,g\in T\leq x;^{\mathfrak{A}}y$. Thus $c\in x;^{\mathfrak{A}}y$. A con tradiction proving $x;^{\mathfrak{A}}y=x;^{\mathfrak{B}}y$. Thus $^2Q\upharpoonleft;^{\mathfrak{A}} = {}^2Q\upharpoonleft;^{\mathfrak{B}}$ and hence $Q\in Su\mathfrak{B}$. Then there is $\mathfrak{G}\subseteq\mathfrak{B}$ with universe Q. Hence

$(*^4)$ $p,q\in\mathfrak{G}\subseteq\mathfrak{B}$ is a standard QRs, that is $\langle\mathfrak{G},p,q\rangle$ satisfies (***).

This is quite obvious, since (***) does hold for $\langle\mathfrak{A},p,q\rangle$ and $\mathfrak{G}\subseteq\mathfrak{A}$. We proved the following

(++) $\langle\mathfrak{A},p,q\rangle$ is a standard QRs $\Rightarrow$ $(\exists\mathfrak{G}\subseteq\mathfrak{B})\langle\mathfrak{G},p,q\rangle$ is a standard QRs, too.

We needed assumption (***) only to prove (++). Therefore in the rest of the proof we do <u>not</u> assume (***).

<u>Claim 2</u> $\mathfrak{B} \notin \mathrm{RA}$, further $\mathfrak{B} \not\models$ "function;function=function".

<u>Proof.</u> $g^{\smile} ;^{\mathfrak{B}} g = g^{\smile} ;^{\mathfrak{A}} g \leq 1'$ (by (III) saying $g \in \mathrm{Fn}\,\mathfrak{A}$) and $h^{\smile} ;^{\mathfrak{B}} h = h^{\smile} ;^{\mathfrak{A}} h \leq 1'$. Hence $h, g \in \mathrm{Fn}\,\mathfrak{B}$. But $h ;^{\mathfrak{B}} g = h ;^{\mathfrak{A}} g = f$. But $f ;^{\mathfrak{B}} f \geq g \nleq 1'$ hence $f \notin \mathrm{Fn}\,\mathfrak{B}$. So "function;function=function" fails in $\mathfrak{B}$. But since this holds in every RA, $\mathfrak{B}$ is not such. (Actually, $f \leq (h;g);g \neq h;(g;g) = h$.) <u>QED(Claim 2)</u>

By Claim 2 and (*) above we proved the following

<u>Claim 3</u> For any choice of $p, q \in \mathfrak{A} \in \mathrm{QRA}$ satisfying (I)-(III) adding the cycle $[f,g,f]$ to the atom-structure of $\mathfrak{A}$ yields a $\mathfrak{B} \in \mathrm{QSA} \sim \mathrm{RA}$ in which composition of functions is not a function in general. <u>In more detail:</u> there is $\mathfrak{U} = \langle U, C, k, I \rangle$ with $f, g \in U$ such that $\mathfrak{A} \cong | \subseteq \mathfrak{Cm}\,\mathfrak{U}$ and $\mathfrak{B} = \mathfrak{Cm}\langle U, C \cup [f,g,f], k, I \rangle \in \mathrm{SA} \sim \mathrm{RA}$ etc. □

By (++) we also proved the following

<u>Claim 4</u> Let $p, q, \mathfrak{A}$ and $\mathfrak{B}$ be as in Claim 3. If $p, q, \mathfrak{A}$ satisfy condition (***), i.e. form a standard QRs, then $(\exists \mathfrak{C} \subseteq \mathfrak{B})$ with $\langle \mathfrak{C}, p, q \rangle$ satisfying (***). I.e. $\mathfrak{B}$ has a standard QRs as a subalgebra. Moreover $\mathfrak{A} \supseteq \mathfrak{C} \subseteq \mathfrak{B}$. □

Since the existence of $\mathfrak{A}, p, q$ satisfying conditions of Claim 4 was indicated below the formulation of (I)-(III), we are done. <u>QED(Thm.3.7.)</u>

<u>REMARK 3.11.</u> (on strengthening Thm.3.7.) (1) As indicated in Claims 3-4, the above proof method is much more general than the theorem it proves. Actually, it appears that every (or

almost every) SA is embeddable into a relativization of a QSA (and also of a standard QSA).: Consider the element $t = (e;1)\cdot(1;e) \in A \subseteq B$. (Within the proof of statement (++), $t = H_0 \times H_0$ was its concrete meaning.) Let $\mathfrak{A}^0 = \mathfrak{Rl}_t\mathfrak{A}$ and $\mathfrak{B}^0 = \mathfrak{Rl}_t\mathfrak{B}$. Then $\mathfrak{A}^0 \in RA$ and $\mathfrak{B}^0 \in SA \sim RA$. The SA $\mathfrak{B}^0$ was obtained from $\mathfrak{A}^0$ by adding a cycle $[f,g,f]$ to its atom-structure. But the proof did not really depend on which cycle was added. The only property we used was that if the new cycle is $[a,b,c]$ then $(\forall\langle r,s,q\rangle \in [a,b,c])\exists u$ with the triple $\langle r,u,q\rangle$ is in the old atom-structure. But this holds whenever $[dom(a)=dom(c)$ & $rng(a)=dom(s)$ & $rng(s)=rng(q)]$ i.e. whenever the new cycle is a "compatible" one. The present proof should carry over to the case when $\mathfrak{B}^0$ is obtained from $\mathfrak{A}^0$ by splitting atoms (below 0′ only), i.e. replacing an old atom u with a set of new atoms having the same domain and range as u. Thus for any SA $\mathfrak{N}$ obtained from an Rs by adding new compatible cycles and splitting atoms is $\subseteq \mathfrak{Rl}_t\mathfrak{B}$ for some standard QSA $\mathfrak{B}$ and $t \in B$ (such that $\mathfrak{Rl}_t\mathfrak{B} \in SA$, too).

(2) The present method seems to be combinable with those in [Ma82]. We start out with the atom-structure $\mathfrak{U}$ for an arbitrary SA $\mathfrak{A}^0$ (i.e. $\mathfrak{A}^0 \subseteq \mathfrak{Cm}\,\mathfrak{U}$). Then for every $b \in U \sim I$ we add a new cycle

$$\begin{tikzcd} & {} \arrow[dr, "b"] & \\ {} \arrow[ur, "p_b"] \arrow[rr, "q_b"'] & & {} \end{tikzcd}$$

with $dom(p_b)=dom(q_b) \notin U$ and $p_b, q_b \notin U$. Then we add new cycles (in a "free generation" manner) to ensure condition (e) of Thm.2.2 of [Ma82]. Repeating this step ω times and checking that each added p_b remains a function (same for q) and letting $p = \sum p_b$, $q = \sum q_b$ we obtain an SA-atom-structure in which $p^{\smile};q=1$. The original $\mathfrak{A}$ is obtained back by relativizing with $\sum U$ (in the new big algebra). This way every SA seems to be embeddable into a relativization of a QSA. For more on these freely generated atom structures see the proof of $WA\subseteq R1RRA$ in [Ma82].$\square$

<u>Proof of Cor.3.9. and Prop.3.10:</u> Let $\mathfrak{B}\in QSA\sim RA$ be as in the proof of Thm.3.7. By [Ma78]Thm.(19),p.150, $SA\subseteq\mathfrak{Ra}^{*}CA_3$. Thus $\mathfrak{B}\subseteq\mathfrak{Ra}\,\mathfrak{C}$ for some $\mathfrak{C}\in CA_3$ generated with $Nr_2\mathfrak{C}$. Since $\mathfrak{B}\in QSA$, (by definition), we have $\mathfrak{C}\in QCA_3$. Since by Claim 4, $\mathfrak{B}$ can be chosen standard, $\mathfrak{C}$ is standard, too (the Q-part of $\mathfrak{C}$, that is). This proves Cor.3.9.

Let $p,q\in\mathfrak{A}\in QRA$ satisfy (I)-(III) in the proof of Thm. 3.7. Instead of adding the cycle $[f,f,g]$, let us add only the triple $\langle f,f,g\rangle$ to C. Let $\mathfrak{B}$ be the algebra obtained this way. Now $\mathfrak{B}\notin SA$ since the Peircean law fails ($f;f\geqslant g$ but $f^{\cup};g\not\geqslant f$). But $\mathfrak{B}\in\mathfrak{Ra}^{*}QCA_3$ can be seen as follows. Let $\mathfrak{A}$ be the set algebra constructed below (I)-(III) for simplicity. Let $\mathfrak{C}\in CA_3$ with $\mathfrak{Ra}\,\mathfrak{C}=\mathfrak{A}$ exists by above quotation as well as by [HMT]§5.2. There is an atom-structure $\mathfrak{U}\in Ca_3$ with $\mathfrak{A}\subseteq\mathfrak{Cm}\,\mathfrak{U}$ and $At\,\mathfrak{A}\subseteq U$ (again incorrectly identifying $At\,\mathfrak{Cm}\,\mathfrak{U}$ with U). We use the method of dilation in 3.2.69 on p.86 of [HMT]Part II, to construct a new CA_3 from $\mathfrak{A}$. We use the notation $a_{\kappa}\in U$ introduced therein. We choose $a_0\leq s_2^0f\cdot\bar{d}(3\times3)$ and $a_1\leq s_2^1f\cdot\bar{d}(3\times3)$ and $a_2\leq g\cdot\bar{d}(3\times3)$. Let n be as therein ($a_0T_0'nT_1'a_1$ etc.) and $\mathfrak{N}$ be the new CA_3 obtained therein (as $\mathfrak{N}=\mathfrak{Cm}\,\mathfrak{U}'$ where $U'=U\cup\{n\}$ etc.). We write "$;^{\mathfrak{N}}$" instead of $;^{\mathfrak{Ra}\mathfrak{C}}$ etc. Then $n\in f;^{\mathfrak{N}}f$ hence by $c_2^{\mathfrak{N}}\{n\}=g$ we have $f;f\geqslant g$ in $\mathfrak{N}$. Since $f\cdot g=0$ and $a_0\leq s_2^0f$, we have $a_0\nleq s_2^0g$, thus $n\notin s_2^{0\mathfrak{N}}g$, thus $n\notin(s_2^1f\cdot s_2^0g)$, hence

$f;g$ is the same in $\mathfrak{N}$ as in $\mathfrak{L}$. Thus $(f^{\cup};g)^{\mathfrak{N}}=f;g\not\geq f$, hence the Peircean law fails in $\mathfrak{N}$. Since the new atom is below $m\overset{d}{=}c_1f\cdot c_0f\cdot s_2^0c_1f$ and m is disjoint from $p,q,p^{\cup},q^{\cup}$ and also from their substituted versions (like s_2^0p) we have that $p^{\cup};q=1$ and $p,q\in Fn\,\mathfrak{Ra}\,\mathfrak{N}$ in $\mathfrak{N}$ ramains true. This proves that $\mathfrak{N}\in QCA_3$ with $\mathfrak{Ra}\,\mathfrak{N}\notin SA$. To see that $\langle\mathfrak{Ra}\,\mathfrak{N}, p,q\rangle$ is standard, the proof of (++) in the above proof of Thm.3.7 (with the obvious modifications) works. QED(Cor.3.9 and Prop.3.10)

We note that dilation can be applied to other non-representable CA's and therefore by modifying the above proof we can extend those non-representable CA_3's to QCA_3's, too. Here a similar remark applies as the one following the proof of Thm.3.7.

REMARK 3.12.(Tarski's problem) Let us return to the problem in §3.10, p.3.78 of [TG]. In 1974 the manuscript of [TG] contained a slightly different version of the problem, namely whether set theory can be formalized in a logic denoted by $\mathcal{L}w^{\times}$ in [TG]p.3.77 which is roughly speaking equivalent with $\overline{\mathrm{Eq}}\mathrm{RA}$'s without associativity, i.e. with the equational theory of Maddux's NA's. By Thm.2(c), the answer is no because both $\overline{\mathrm{Eq}}\mathrm{NA}$ and $\overline{\mathrm{Eq}}\mathrm{WA}$ are decidable. Of course, it was already known (following Maddux's result $NA\supsetneqq WA\supsetneqq SA\supsetneqq RA$) that the two systems $\mathcal{L}w^{\times}$ and $\mathcal{L}w^{\times}$ are not equipollent. What is new here is that in the weaker system there is no way of formalizing set theory. Motivated by Maddux's result, the problem in [TG]§3.10 was changed and the new question asks if set theory can be formalized in a logic denoted by $\mathcal{L}\omega^{x}$ on p.3.77 of [TG] which is equivalent with $\overline{\mathrm{Eq}}$SA or almost equivalently in our version of L_3 (i.e. in $\overline{\mathrm{Eq}}\mathrm{CA}_3$). Further, it asks if the main objectives of [TG] can be carried through in $\overline{\mathrm{Eq}}$SA. The question is complex, and so is the answer.

(1) Positive answer: By Prop.3.3, there is a computable $\kappa : Fm_{\omega}^{2} \to Fm_3$ with $(\forall\varphi \in Fm_{\omega}^{2})[Ax \models \varphi \iff Ax \vdash_{3} \kappa\varphi]$. Hence full first-order logic Fm_{ω}^{0} can be simulated in our L_3 (hence in $\overline{\mathrm{Eq}}$SA as well as in $\overline{\mathrm{Eq}}\mathrm{CA}_3$) via the translation mapping κ. To be precise, any theory of full first-order logic containing Ax can be simulated in L_3 (hence in $\overline{\mathrm{Eq}}$SA, $\overline{\mathrm{Eq}}\mathrm{CA}_3$). Since set theory does contain Ax it can be built up in L_3 this way. Since this was the main aim of §4 in [TG] and this was the central question in the quoted problem in §3.10 of [TG], we have a positive answer. However, when using this simulated set theory (in our L_3) one has to be careful to use only those formulas which are in Rgκ. This aesthetic shortcoming cannot be eliminated by Thm.3.7 above. This leads to the negative answer.

Before turning to the negative answer, we have to say more about the positive one. In §3.7 of [TG] Tarski's original $\mathcal{L}_3$ is discussed which is weaker than that version of $\mathcal{L}_3$ which is adopted in the rest of [TG]. The original $\mathcal{L}_3$ is also much simpler, and therefore the problem is indicated in §3.7 if formalization of set theory could be carried through in this simpler original $\mathcal{L}_3$. Since this original $\mathcal{L}_3$ is slightly stronger than our L_3 i.e. than $\overline{\mathrm{Eq}}\mathrm{CA}_3$, the answer is positive, by Prop.3.3 (as indicated above). Thus the problems in §3.7 of [TG] also receive a kind of a positive answer (at least in a sense i.e. modulo formalizability of set theory).

Further, using the terminology of §4 of [TG], there is an equipollence between $\langle Rg\mathcal{K}, \vdash_3 \rangle$ (i.e. a subset of Tarski's original $\mathcal{L}_3$) and $\mathcal{L}$ relative to the axiom Ax , that is relative to any strong pairing axiom. (Our π corresponds to Q_{AB} of [TG] and our Ax could be denoted as Q^+_{AB} to indicate $Q^+_{AB} = (Q_{AB}$ + some further facts true for real pairing functions).) Recall that $Rg\mathcal{K} \subseteq Fm^1_3$ hence $\langle Rg\mathcal{K}, \vdash_3 \rangle$ is a subsystem of our L_3. Imitating the style of Thm.($x\ell$) on p.4.50 of [TG], we have

$$(\forall\varphi\in Rg\mathcal{K})\left[\vdash_{Q^+_{AB},3}\varphi \iff \models_{Q^+_{AB}}\varphi\right].$$

In our notation this is Prop.3.3(iv). The above is a completeness theorem for the logic $\langle Rg\mathcal{K} , Ax\vdash_3 \rangle$.

(2) Let $\vdash_{3+}$ be provability in "(L_3 + RA-axiom-schemes)". This has the same power as $\overline{\mathrm{Eq}}$RA (using the terminology of [TG], $\vdash_{3+}$ and $\overline{\mathrm{Eq}}$RA are equipollent). On p.4.48, Thm.(xxxvi) of [TG] proves a stronger version of our Prop.3.3(i) for $\vdash_{3+}$, namely Prop.3.3$^+$ says $Ax\vdash_{3+} (\varphi \leftrightarrow \mathcal{K}\varphi)$ for all $\varphi\in Fm^2_3$. This implies that all sentences of L_3 can be used when formalizing set theory with this stronger system $\vdash_{3+}$. This result fails for $\vdash_3$, moreover it cannot be recovered by adding more axioms on the pairing functions to Ax as Prop. 3.5 + Thm.3.7(i) in the present work show. Namely, there is a standard $\mathfrak{A}\in QSA\sim RA$ (that is in $\mathfrak{A}$ all possible postulates (true in the standard model) about p and q are satisfied).

The relative equipollence of $\mathcal{L}^{+}$ and $\mathcal{L}^{\times}$ stated on p.4.49 of [TG] (below (xxxvii)) implies equipollence of $\mathcal{L}(=L_{\omega})$ with $\mathcal{L}_{3}^{+}$ relative to pairing axioms Q_{AB} i.e. relative to our π. This equipollence does not carry over to $\mathcal{L}w^{\times}$ (occurring in the problem in §3.10 of [TG]) in place of $\mathcal{L}_{3}^{+}$. (Recall that $\mathcal{L}w^{\times}$ is slightly stronger than our L_3 which in turn is slightly weaker than Tarski's original weak $\mathcal{L}_3$ in §3.7 of [TG]. $\mathcal{L}w^{\times}$ is equivalent with $\overline{\mathrm{Eq}}$SA.) $\mathcal{L}$ is not equipollent with $\mathcal{L}w^{\times}$ relative any Q_{AB}. Moreover, no matter how strong pairing axioms Q_{AB}^{++} $(\supseteq Q_{AB}^{+})$ we choose, $\mathcal{L}$ will not be equipollent with $\mathcal{L}w^{\times}$ relative to Q_{AB}^{++} . More precisely, if Q_{AB}^{++} is any set of sentences about projection functions A and B such that Q_{AB}^{++} is true in the standard model $\langle H_{\omega},A,B\rangle$ of projections (where $A\langle a,b\rangle=a$ and $B\langle a,b\rangle=b$ for any $a,b\in H_{\omega}$) as described in Def.3.8(iii) above, then $\mathcal{L}$ is <u>not</u> equipollent with $\mathcal{L}w^{\times}$ relative to Q_{AB}^{++}. This follows from Thm.3.7(i) above. It might be an interesting contrast with this to recall that the subformalism $\langle \mathrm{Rg}K, \mathrm{Ax} \vdash_{3}\rangle$ is equipollent with $\mathcal{L}$ relative to Q_{AB}^{+} , as observed in item (1) above.

(3) It was brought to our attention by Roger Maddux that Tarski used his "translation mapping theorem" (between L_{ω} and $\overline{\mathrm{Eq}}$QRA) to represent first-order theories (containing Ax) as QRA's in such a fashion that every QRA represents some theory and finite theories correspond to finitely presented QRA's. By the present results, the same can be done by Q^{+}SA's, too. In one direction this is not surprising (since QRA $\subseteq$ QSA) but despite of the negative Thm.3.7 above, every Q^{+}SA represents some theory. Hint: Let $\mathfrak{A}\in$ QSA. Then there is a surjective homomorphism $k:\mathfrak{F}\twoheadrightarrow\mathfrak{A}$ for some free SA $\mathfrak{F}$. Select

$\bar{p},\bar{q}\in F$ as some pre - k - images of p,q of $\mathfrak{A}$. It is not hard to translate Ora (the universe of $\mathfrak{Ora}$ based on $\bar{p},\bar{q}$) such that it becomes a generalized reduct of the free SA $\mathfrak{F}$. Let us project (this new) Ora into $\mathfrak{A}$ along k. Then k^{*} [illegible] will be a generalized reduct of $\mathfrak{A}$. Since the operations of (the new) $\mathfrak{Ora}$ are defined in terms of the operations of $\mathfrak{F}$ which in turn are preserved by k we have $k^{*}(\mathfrak{Ora})$ as a generalized subreduct of $\mathfrak{A}$. But since $\mathfrak{A}\in QSA$ and $\langle k(\bar{p}), k(\bar{q})\rangle=\langle p,q\rangle$ we conclude that $k^{*}(\mathfrak{Ora})$ is a QRA, hence representable. Now the theory corresponding to $\mathfrak{A}$ is obtained via $Fm^{0}_{\omega}\xrightarrow{\kappa} Ora \xrightarrow{k} A$. Namely, $T=\{\varphi\in Fm^{0} : k\kappa\varphi=k\kappa(\underline{T})\}$ is the theory connected to $\mathfrak{A}$. The same can be done for $Q^{+}CA_{3}$ in place of $Q^{+}SA$. For more concrete information on the subject matter of the present item (3) see Remark 3.13 below.$\square$

<u>REMARK 3.13.</u> We know that every CA_{4} $\mathfrak{A}$ has an RA as a reduct $\mathfrak{Ra}\,\mathfrak{A}$ and we also know that for some $\mathfrak{B}\in CA_{3}$, $\mathfrak{Ra}\,\mathfrak{B}\notin RA$ (see e.g. [Ma78]). One might hope that the RA-reducts of $Q^{+}CA_{3}$'s would be RA's, but this is not true by Thm.3.7, namely $\mathfrak{Ra}\,\mathfrak{B}\notin RA$ for some $Q^{+}CA_{3}$ $\mathfrak{B}$ (moreover $Q^{+}SA\notin RA$). Despite of all these negative results, there is a way to associate a generalized reduct to every $Q^{+}CA_{3}$ which will always be an RA. This goes by translating the definition of $\mathfrak{Ora}$ in Fm_{3} (see §2) into the language of CA_{3}. In §2 after proving L.2.7, we defined $\mathfrak{Ora}/\equiv_{Ax}$ as a generalized reduct of the CA_{3} $\mathfrak{Fm}^{\wedge}/\equiv_{Ax}$. More precisely, in Def.2.9 we defined operations $\odot$, $^{\smile}$, 1, $1'$ on the set Fm^{1}_{3}. Later these became the operations of the QRA $\mathfrak{Ora}/\equiv_{Ax}$. By using the translation $\tau\mu : Fm_{3}\to$ "terms of CA_{3}"

introduced in §4.3 of [HMT]Part II p.171, we translate the definitions of $\odot$, $\overset{\cdot}{\smile}$, $\dot{1}$, $\dot{1}'$ into cylindric terms. E.g. we obtain the cylindric term defining $v_0 \underline{\odot} v_1$ by letting $v_0 \underline{\odot} v_1 \overset{d}{=} \mathfrak{M}(R_0(v_0v_1v_2) \odot R_1(v_0v_1v_2))$ where of course the occurrence of $\odot$ on the right side should be replaced by the formula in Def.2.9 defining it. So we obtained four CA-terms $\underline{\odot}$, $\underline{\overset{\cdot}{\smile}}$, $\underline{\dot{1}}$, $\underline{\dot{1}'}$ Let $\mathfrak{A}$ be a Q^+CA_3 with projections $p,q \in Fn\,\mathfrak{A}$. Of course the cylindric terms $\underline{\odot}$ etc. contain two parameters p and q, which now in $\mathfrak{A}$ can be fixed. So after fixing p,q, $\underline{\odot} = \underline{\odot}^{\mathfrak{A}} : {}^2A \rightarrow A$, $\underline{\dot{1}} = \underline{\dot{1}}^{\mathfrak{A}} \in A$ etc. We define $\mathfrak{Ruj}\mathfrak{A} \overset{d}{=} \langle \mathfrak{Rl}_{\underline{1}} \mathfrak{Bl}\mathfrak{A}, \underline{\odot} , \underline{\overset{\cdot}{\smile}} , \underline{\dot{1}'} \rangle$. Then we can apply a slight generalization of Prop.2.10 (see Prop.3.3 for generalizing to many generators) to prove that $\mathfrak{Ruj}\mathfrak{A} \in RA$. (Since we started out with a Q^+CA_3, we can prove $\mathfrak{Ruj}\mathfrak{A} \in QRA$, too, and hence by Tarskiś representation theorem, $\mathfrak{Ruj}\mathfrak{A} \in QRRA$). We have outlined how to prove that for any Q^+CA_3 its generalized reduct $\mathfrak{Ruj}\mathfrak{A}$ is an RA. ☐

LIST OF (SPECIAL) SYMBOLS

symbol	page	symbol	page	symbol	page
L_α	.(V)	SA, WA, NA	.20,21	h	.55
$\Lambda=\langle\alpha,\ell,\varrho\rangle$	.1	$\overline{EqK}$	.21	$R(x_0x_1)$	.55
Fm^Λ	.1	$\mathfrak{Fr}_\beta(\Delta)_K$	.[HMT]	$\mathfrak{R}$	.62
$\underline{T},\underline{F}$	.1	(C_0-C_7)	.[HMT]	$m^{\mathfrak{M}}$	.62
restricted	.1	E	.22	$\mathcal{K}$	.63
$\models_\alpha$	.2	Λ_α , Fm_α	.22,23	$\mathcal{K}'$	.63
$\vdash_{r,\Lambda}$	.3	x,y,z	.23	$\mathfrak{K}$	.63
$\vdash_{r}$	.3	(S)	.23	π'_{RA}	.63
$\vdash_\alpha$	.3	Fm^H_α , Fm^2_3	.24	$h'(\pi'_{RA})$	.63
$\vdash_{r,\alpha}$	.3	$p_i(x,y)$	.24	$\models_{Ax}$	.63
Λ_f^Λ	.3	π	.24	$\vdash_{Ax}$	.63
((1)) - ((9))	.3	f	.25	Bq_α	.70
(MP), (G)	.3	Rs	.27	$\tau\mu$	.71
monadic	.4	SimRA	.27	$Fn\mathfrak{A}$	.72
$MGR(\varphi)$	.5	$RAT_{\mathfrak{X}}$, RAT	.27	standard	.73
$Fm^{\Lambda,0}$	.8	$\tau^{\mathfrak{A}}(X)$	.27	QSA, QCA_3	.73
$\hat{T}$	.8	base$\mathfrak{A}$	.27	Q^+SA, Q^+CA_3	.73
G.i.	.8	r	.28	H_i , H_ω	.73,75
w.G.i.	.8	$s^i_j\varphi$	.28	dom(x)	.74
$\mathfrak{Fm}^\Lambda$, $\mathfrak{Fm}^{\Lambda,0}$	.11	inseparable	.32	rng(x)	.74
$p^{\equiv^\Lambda}$	.11	p,q	.33	0'	.74
$p^{\mathfrak{Fm}^\Lambda}$, $p^{\mathfrak{Fm}^{\Lambda,0}}$	.11	π_{RA}	.33	$\mathfrak{Lm}\,\mathfrak{A}$	.75
$\Delta^{\mathfrak{A}}(x)$	.14	η	.33	[f,g,f]	.76, [Ma82
$Zd\mathfrak{A}$	.14	π'	.34	$\mathfrak{Ra}\,\mathfrak{A}$	.[HMT]
$\mathcal{M}^\Lambda$, $\models^{CA}_\alpha$	.15	$\mathcal{G}$	.36	Q_{AB}	.85
C(1)-C(8)	.16	$x_i=y_j$	.38	Q^+_{AB}	.85
$\mathfrak{Ls}^{\mathfrak{m}}$, $\tilde{\varphi}^{\mathfrak{m}}$	.18	φu_i	.38	Q^{++}_{AB}	.86
CA_α	.20	$\mathfrak{Ora}$, Ora	.39		
$\mathfrak{Zd}\mathfrak{A}$, $Zd\mathfrak{A}$	.20	pair(x)	.39		
$\mathfrak{Fr}_\beta K$	.20	$\odot,{}^{\odot},\grave{\imath},\grave{\imath}',\dot{-}$	.39		
$\mathfrak{Rf}(K,\alpha)$	.20	$\equiv_{Ax}$, $\mathfrak{Fm}_\alpha$	.40		
$\mathcal{R}(U)$	.20	Ax	.41		
IGs_α	.20	"by FO"	.42		
RA, RRA	.20	$\Delta(u,v)$	.43		

REFERENCES

[AGN77] Andréka,H. Gergely,T. Németi,I., On universal algebraic construction of logics. Studia Logica 36(1977)1-2, 9-47.

[AN80] Andréka,H. Németi,I., On systems of varieties definable by schemes of equations. Algebra Universalis 11(1980)105-116.

[AN81] Andréka,H. Németi,I., Dimension complemented and locally finite dimensional cylindric algebras are elementarily equivalent. Algebra Universalis 13(1981) 157-163.

[B85] Biró,B., personal communication.

[Bu85] Burmeister,P., A model theoretic approach to partial algebras. Akademie Verlag, Berlin, 1985.

[CT51] Chin,L.H. Tarski,A., Distributive and modular laws in the arithmetic of relation algebras. Univ. of California Publ. in Math., new series 1,9(1951) 341-384.

[E72] Enderton,H.B., A mathematical introduction to logic. Academic Press, New York, 1972.

[H67] Henkin,L., Logical systems containing only a finite number of symbols, Séminaire de mathématiques supérieures, no 21, Les Presses de l'Université de Montréal, Montréal, 1967. 48pp.

[H73] Henkin,L., Internal semantics and algebraic logic. Truth, syntax, and modality, North-Holland, 1973. 111-127.

[H83] Henkin,L., Proofs in first-order logics with only finitely many variables. Abstr. Amer. Math. Soc., vol 4, 1983. p.8.

[HMT] Henkin,L. Monk,J.D. Tarski,A., Cylindric Algebras. Parts I-II. North-Holland, Amsterdam, 1971, 1985.

[HMTAN] Henkin,L. Monk,J.D. Tarski,A. Andréka,H. Németi,I., Cylindric Set Algebras. Lecture Notes in Math. 883, Springer-Verlag, Berlin, 1981.

[Johnson73] Johnson,J.S., Axiom systems for logic with finitely many variables. J. Symbolic Logic 38(1973) 576-578.

[J82] Jónsson,B., Varieties of relation algebras. Algebra Universalis 15(1982) 273-298.

[J84] Jónsson,B., The theory of binary relations. First draft. Preprint 1984. 65 pp.

[JT52] Jónsson,B. Tarski,A., Boolean algebras with operators. Part II. Amer. J. Math. 74(1952) 127-162.

[Ma78] Maddux,R., Topics in relation algebras. Doctoral Dissertation. Univ. of California, Berkeley, 1978. iii+241pp.

[Ma78a] Maddux,R., Some sufficient conditions for the representability of relation algebras. Algebra Universalis 8(1978) 162-171.

[Ma80] Maddux,R., The equational theory of CA_3 is undecidable. J. Symbolic Logic 45(1980) 311-316.

[Ma82] Maddux,R., Some varieties containing relation algebras. Trans. Amer. Math. Soc. 272(1982) 501-526.

[Ma83] Maddux,R., A sequent calculus for relation algebras. Ann. Pure App. Logic 25(1983) 73-101.

[Ma84] Maddux,R., Finite integral relation algebras. Proc. Conf. Universal Algebra and Lattice Theory (Charleston, 1984), to appear.

[M61] Monk,J.D., Studies in cylindric algebras. Doctoral dissertation, Univ. of California, Berkeley, 1961. vi+83 pp.

[M61a] Monk,J.D., Relation algebras and cylindric algebras. Notices Amer. Math. Soc. 8(1961) p.358.

[M62] Monk,J.D., Singulary cylindric and polyadic equality algebras. Trans. Amer. Math. Soc. 112(1964) 185-205.

[M69] Monk,J.D., Nonfinitizability of classes of representable cylindric algebras. J. Symbolic Logic 34(1969) 331-343.

[M71] Monk,J.D., Provability with finitely many variables. Proc. Amer. Math. Soc. 27(1971) 353-358.

[M76] Monk,J.D., Mathematical Logic. Springer Verlag, 1976.

[M77] Monk,J.D., Some problems in algebraic logic. Colloq. Inter. de Logic, CNRS 249(1977) 83-88.

[N78] Németi,I., Connections between cylindric algebras and initial algebra semantics of CF languages. In: Mathematical Logic in Computer Science (Proc. Salgotarján 1978) Colloq.Math.Soc. J.Bolyai 26, North-Holland, 1981. 561-605.

[N80] Németi,I., Some constructions of cylindric algebra theory applied to dynamic algebras of programs. CL&CL 14(1980)43-65.

[N84] Németi,I., Algebraic proofs of "$\mathfrak{Fr}_\beta \mathrm{CA}_\alpha$ ($\alpha \geqslant 4$) is not atomic" Manuscript, 1984.

[N84a] Németi,I., On infinite dimensional free cylindric algebras. Preprint, 1984.

[N85] Németi,I., Free cylindric and relation algebras are not atomic. Abstracts of Oberwolfach meeting "Boolean Algebras", 1985.

[N85a] Németi,I., Not every relation algebra is a reduct of CA_4. Manuscript, 1985.

[N85b] Németi,I., Exactly which varieties of cylindric algebras are decidable? Preprint no 34/85, 1985.

[N85c] Németi,I., Decidability of relation algebras with weakened associativity. Preprint, 1985.

[Poizat82] Poizat,B., Deux ou trois choses que je sais de L_n. J. Symbolic Logic 47(1982) 641-658.

[Sain82] Sain,I., Finitary logics of infinitary structures are compact. Abstracts Amer. Math. Soc. April 1982, Issue 17, Vol 3, No 3, p.252.

[Schröder85] Schröder,E., Note über die Algebra der binären Relative. Math. Ann. 46(1895) 144-158.

Shepherdson,J.C., Variants of Robinson's essentially undecidable theory. Arch. Math. Logic Grundlag. 23(1983)1-2, 61-64. MR 85d:03088

[T41] Tarski, A., On the calculus of relations. J. Symbolic Logic 6(1941) 73-89.

[T53] Tarski, A., A formalization of set theory without variables. J. Symbolic Logic 18(1953) p.189.

[T53a] Tarski, A., Some metalogical results concerning the calculus of relations. J. Symbolic Logic 18(1953) 188-189.

[TG] Tarski, A. Givant, S., A formalization of set theory without variables. Manuscript, cca 500pp.

Mathematical Institute of the
Hungarian Academy of Sciences
Budapest, Pf. 127,
H-1364 Hungary