Irreducible transformations on the space of t-uniform hypergraph realizations of degree sequences

Research proposal, 2026 Spring

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1 Problem description

A hypergraph $H = (V, E)$ is a generalization of graphs, where $E(H) \subseteq 2^V \setminus \emptyset$, that is, a hyperedge is a non-empty subset of the vertices. Clearly, a simple graph is a hypergraph where each hyperedge is a subset of vertices of size 2. A hypergraph is a *t-uniform hypergraph* if each hyperedge is a subset of size t (that is, simple graphs are 2-uniform hypergraphs). The *degree* of a vertex is the number of hyperedges incident with it. A degree sequence D is a finite series of non-negative integers. A hypergraph H is a *realization* of D if the degrees of the vertices of H equal the numbers in D. If a realization of D exists, we say that D is *hypergraphic* or simply *graphic*. A degree sequence may have many different hypergraph realizations. A natural question is whether these realizations are isolated objects, or whether they can be transformed into one another by simple local operations.

A *switch operation* on a 3-uniform hypergraph $H = (V, E)$ removes two hyperedges $(a_1, b_1, c_1), (a_2, b_2, c_2) \in E(H)$ and adds two hyperedges (a_2, b_1, c_1) and (a_1, b_2, c_2) that were not in H before the switch operation. It is easy to see that switches do not change the degrees of H. We are interested in degree sequence classes for which any realization of a degree sequence D in that class can be transformed into any other realization of D with a finite series of switch operations. In such a case, we say that the switches are *irreducible* on the space of realizations. Such sufficient transformations are crucial for sampling realizations of a degree sequence. Sampling hypergraph realizations of a degree sequence is a key task in modern statistics [5].

It is well-known that (the analogous) switch operations are irreducible on the space of simple graph (that is, 2-uniform hypergraph) realizations of any degree sequence [4, 3]. However, it is also known that switches are not irreducible on the space of 3-uniform hypergraph realizations of some degree sequences [5]. This raises the natural question of characterizing those degree sequences for which the space of t-uniform hypergraph realizations is connected under suitably defined local operations, and to identify minimal sets of transformations that guarantee irreducibility.

Somewhat paradoxically, even deciding whether a 3-uniform hypergraph realization of a degree sequence exists is already an NP-complete problem in general [1]. This makes it far from obvious that the space of realizations should admit simple transformation rules. However, there are special degree classes for which it is easy to decide if a 3-uniform hypergraph exists with those given degrees. Even more notably, we can give degree sequence classes which are *fully graphic*, that is, any degree sequence in that class is graphic. The main result of the 2024 Spring reseach class was to characterize those (c_1, c_2) pairs for which each degree sequence of length n with sum of degrees divisible by 3 and all degrees between $c_1\binom{n-1}{2}$ and $c_2\binom{n-1}{2}$ is graphic (technical remark: there might be finite number of exceptions). Furthermore, the bounds obtained are strict in the sense that out of these bounds the degree sequence problem becomes NP-complete [8]. To conclude, the parametric 3-uniform hypergraphic degree sequence problem is either extremely easy (one just has to check if the sum of the degrees can be divided by 3) or hopelessly hard (NP-complete).

In a follow-up research, we also proved a strict dichotomy theorem for sparse t-uniform hypergraphs [9]. Observe that $\binom{n-1}{t-1} = \Omega(n^{t-1})$. Therefore, a degree sequence is considered sparse if its maximal degree is $o(n^{t-1})$ (for those who are unfamiliar with the $o()$ notation, it roughly means much smaller than n^{t-1}). We considered degree sequences between $n^{\alpha'}$ and $6n^{\alpha}$, $0 \leq \alpha' \leq \alpha < t-1$. It turns out that the hypergraphicality problem with this parameterization either can be solved in linear time or it is NP-complete. Again, the parametric regime where the realization problem can be solved in linear time is a fully graphic degree sequence class.

These results naturally suggest that the dichotomy observed for the *existence* of realizations may also manifest itself at the level of *reconfiguration*. In particular, it is tempting to conjecture that fully graphic degree sequence classes – especially those parameterized by lower and upper bounds on the degrees – admit irreducible switch operations on their space of realizations. In other words, whenever realizability is guaranteed based on purely minimum and maximum degrees, the corresponding realization space may also be connected under simple local transformations. Establishing irreducibility in such parametric regimes, or identifying precise obstructions to it, is a central motivation of the proposed research.

The suggested research problems for the 2026 Spring semester are the following:

1. **Transforming realizations into each other.** The primary goal of the project is to study the connectivity of the space of 3-uniform hypergraph realizations of a fixed degree sequence under switch operations. Given a degree sequence D, we ask whether any two realizations of D can be transformed into each other by a finite sequence of switches, and if not, to identify explicit obstructions to such transformations.

2. **Irreducibility in parametric degree classes.** Motivated by recent dichotomy results for the existence of realizations, we aim to investigate degree sequence classes parameterized by minimum and maximum degree bounds for which realizability is guaranteed. A central question is whether these fully graphic classes also admit irreducible switch operations, or whether additional local transformations are necessary to ensure connectivity.

3. **Minimal transformation rules.** If standard switch operations are not sufficient to achieve irreducibility, an important direction is to design and analyze natural extensions

of switch operations. The goal is to determine small and intuitive sets of local moves that connect all realizations within a given parametric degree class.

4. **Transforming linear hypergraphs into each other** A hypergraph is *linear* if any two hyperedges intersect in at most one point (so the hyperedges have the same property as lines in geometry). We are interested in when a degree sequence has a linear graph realization, and further how the realizations can be transformed into each other. There are some preliminary results. A *Latin square* is an $n \times n$ table filled by numbers from $\{1, 2, \ldots, n\}$ such that each number is presented in each row and each column exactly once. Any $n \times n$ Latin square can be unequivocally represented by a 3-uniform hypergraph of $3n$ vertices, where the vertices are the n rows, the n columns and the first n positive integers, and there is a hyperedge (a, b, c) if number c is in row a and column b. Such a hypergraph is a *tripartite* hypergraph, that is, there are 3 disjoint vertex sets of the vertices such that each hyperedge contains exactly one vertex from each vertex class. In addition, such a hypergraph is *linear*. It is known that any $n \times n$ Latin square can be transformed into any other $n \times n$ Latin square via Latin squares by perturbing at most 3 rows in each step [6]. These transformations create larger perturbations than switches, however, such large perturbations are necessary for irreducibility. Also, it is worth mentioning that these larger perturbations not only maintain the underlying degree sequence but also the linear property of the intermediate tripartite hypergraphs.

2 Qualifying problems

Please, solve the first 3 of the following exercises and at least one of exercises 4 and 5. Solutions to the remaining exercises will be considered in case of competition. Solving both exercises 4 and 5 is also considered in case of competition.

1. Show that the sum of the degrees in a t-uniform hypergraph must be divisible by t. Give an example of a degree sequence D of length 5 in which the sum is divisible by 3, each degree is positive, the maximum degree is at most $6 \left(= \binom{5-1}{3-1}\right)$, but D does not have a 3-uniform hypergraph realization.

2. A hypergraph as well as its corresponding degree sequence are called *regular* if all degrees are the same. Prove that regular tripartite degree sequences on $n + n + n$ vertices are graphic if the common degree is at most n^2.

3. Construct two 5×5 Latin squares such that the first one cannot be transformed into the second one by a series of transformations each affecting only two rows or two columns. Any intermediate object obtained by the transformations must be a Latin square, too.

4. Prove that any realization of a tripartite regular degree sequence D can be transformed into any other realization of D by a series of switch operations. If you are planning to

solve this exercise, please also read exercise 5. In your proof, highlight where you use the condition of regularity. Observe that the statement does not hold for an arbitrary degree sequence (otherwise exercise 5 would not have a solution), so you have to use regularity somewhere.

5. Find two tripartite hypergraphs H_1 and H_2 with the same degree sequences such that H_1 cannot be transformed into H_2 by a series of switch operations.

6. A *Latin rectangle* is a $k \times n$ table, $k < n$, such that each row contains the numbers from 1 to n, and there is no column that contains the same number twice. It is a known theorem that any Latin rectangle can be extended into a Latin square. Based on this known theorem, prove that for each prime $p > 3$, there are two $p \times p$ Latin squares such that the first one cannot be transformed into the second one by a series of transformations each affecting only two rows or two columns. Any intermediate object obtained by the transformations must be a Latin square, too.

7. **(Research-level problem)** A *hinge-flip* operation on a 3-uniform hypergraph $H = (V, E)$ removes a hyperedge (a_1, b, c) and adds a hyperedge (a_2, b, c) that was not in $E(H)$ before the hinge-flip operation. Note that a hinge flip decreases the degree of a_1 by 1 and increases the degree of a_2 by 1. Also, it does not change the degree of any other vertex. It also does not change the sum of the degrees. Find a class of degree sequences $\mathcal{D}$ with the following properties. Any $D = (d_1, d_2, \ldots, d_n) \in \mathcal{D}$ and any 3-uniform realizations H_1 and H_2 of D, H_1 can be transformed into H_2 with a sequence of hinge-flips and switch operations such that any intermediate hypergraph H' has a degree sequence $D' = (d'_1, d'_2, \ldots, d'_n)$ satisfying

$$\sum_i |d_i - d'_i| \leq 2.$$

Note that there are multiple solutions to this problem, try to find as nice solution as possible. Solving this problem is not required for qualification, but strong partial progress will be considered positively.

References

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