Parametric complexity of 3-uniform hypergraph problems

Research proposal, 2025 Summer

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1 Problem description

A hypergraph $H = (V, E)$ is a generalization of graphs, where $E(H) \subseteq 2^V \setminus \emptyset$, that is, a hyperedge is a non-empty subset of the vertices. Clearly, a simple graph is a hypergraph where each hyperedge is a subset of vertices of size 2. A hypergraph is a *k-uniform hypergraph* if each hyperedge is a subset of size k (that is, simple graphs are 2-uniform hypergraphs). The *degree* of a vertex is the number of hyperedges incident with it. We denote the degree of v by $d(v)$. A degree sequence D is a finite series of non-negative integers. A hypergraph H is a *realization* of D if the degrees of the vertices of H equal the numbers in D. If a realization of D exists, we say that D is *hypergraphic* or simply *graphic*.

Surprisingly, the problem of whether there is a 3-uniform hypergraph with prescribed degrees is already an NP-complete problem [1]. However, there are special degree classes for which it is easy to decide if a 3-uniform hypergraph exists with those given degrees. Even more notably, we can give degree sequence classes which are *always graphic*, that is, any degree sequence in that class is graphic. The main result of the 2024 Spring reseach class was to characterize those (c_1, c_2) pairs for which each degree sequence of length n with sum of degrees divisible by 3 and all degrees between $c_1\binom{n-1}{2}$ and $c_2\binom{n-1}{2}$ is graphic (technical remark: there might be finite number of exceptions). Further, the obtained bounds are strict in the sense that out of these bounds the degree sequence problem becomes NP-complete [5].

To conclude, the parametric 3-uniform hypergraphic degree sequence problem is either extremely easy (one just has to check if the sum of the degrees can be divided by 3) or hopelessly hard (NP-complete). We are interested in further parametrized problems in algorithmic hypergraph theory, and would like to find algorithmic problems that are easy to solve in certain parameter regime. The suggested research problems are the following:

1. **Transforming realizations into each other** A *switch operation* on a 3-uniform hypergraph $H = (V, E)$ removes two hyperedges $(a_1, b_1, c_1), (a_2, b_2, c_2) \in E(H)$ and adds two hyperedges (a_2, b_1, c_1) and (a_1, b_2, c_2) that were not in H before the switch operation. It is easy to see that switches does not change the degrees of H. We are interested in degree sequence classes for which any realization of a degree sequence D in that class can be

transformed into any another realization of D with a finite series of switch operations. The above-mentioned always graphic degree sequence classes or a subset of them seems to be a good candidate.

2. **Transversals in Latin squares and matchings in tripartite hypergraphs** A *Latin square* is an $n \times n$ table filled by numbers from $\{1, 2, \ldots, n\}$ such that each number is presented in each row and each column exactly once. A *transversal* is a selection of cells with no common rows, columns or numbers. It is a long-standing conjecture (called the Ryser-Brualdi-Stein conjecture) that for any odd n, any $n \times n$ Latin square contains a transversal of size n. A slightly weaker version for even n's are recently proved [6]. Any $n \times n$ Latin square can be unequivocally represented by a 3-uniform hypergraph of $3n$ vertices, where the vertices are the n rows, the n columns and the first n positive integers, and there is a hyperedge (a, b, c) if number c is in row a and column b. Such a hypergraph is a *tripartite* hypergraph, that is, there are 3 disjoint vertex sets of the vertices such that each hyperedge contains exactly one vertex from each vertex class. In addition, such a hypergraph is *linear* which means that there are no two hyperedges that share more than one vertex. A transversal is a *matching* in this hypergraph.

   It is known that any $n \times n$ Latin square can be transformed into any another $n \times n$ Latin square by perturbing at most 3 rows in each step [3]. If such perturbations are possible with maintaining a full transversal (a transversal of length of n) for any odd n, then we could prove the Ryser-Brualdi-Stein conjecture. We are also interested in tripartite, linear 3-uniform hypergraph classes with guaranted transversals (that is, matchings) of given size.

2 Some further background

- The key concept seems to be the *hinge flip operations* and their relation to the graphicality of degree sequences. Please read our manuscript (`https://arxiv.org/pdf/2312.00555.pdf`) from the beginning till Theorem 2.2., and make sure you have understood the concept of hinge flips.

- Consider the neighborhood of a vertex in a 3-uniform hypergraph. It is a set of pair of vertices, and we can look at it as the edge set of a simple graph. This explains why simple graphs and their degree sequences are considered in the research of 3-uniform hypergraphs.

  The Erdős-Gallai theorem on the graphicality of simple graph degree sequences might also play a central role in our research that we state here.

  Theorem 1. *[2] Let $D := d_1 \geq d_2 \geq \ldots \geq d_n$ be a degree sequence. Then G is graphic if and only if*

1. $\sum_{i=1}^n d_i$ *is even and*

2. for all $k = 1, 2, \ldots n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{d_j, k\}.$$

- The Tripathy-Vinay theorem improves the Erdős-Gallai theorem in the sense that it shows that is is sufficient to check only a certain subset of the inequalities [7]. Most notably, assume that a degree sequence consists of k number of degree $d_{\max}$ and $n-k$ number of degree $d_{\min}$. Then it is graphic if and only if the sum of the degrees is even and

  $$kd_{\max} \leq k(k-1) + (n-k)\min\{k, d_{\min}\}.$$

  Furthermore, if a degree sequence contains k number of degree $d_{\max}$ and $n-k-1$ number of degree $d_{\min}$ and exactly one degree d_x, $d_{\min} < d_x < d_{\max}$, then it is graphic if and only if the sum of the degrees is even and the following two inequalities hold

  $$kd_{\max} \leq k(k-1) + (n-k-1)\min\{k, d_{\min}\} + \min\{k, d_x\},$$

  $$kd_{\max} + d_x \leq (k+1)k + (n-k-1)\min\{k, d_{\min}\}.$$

3 Qualifying problems

Please, solve at least the first 4 of the following exercises. Solutions to the remaining exercises will be considered in case of competition.

1. A hypergraph as well as its corresponding degree sequence are called *regular* if all degrees are the same. Prove that regular tripartite degree sequences on $n+n+n$ vertices are always graphic if the common degree is at most n^2.

2. Prove that any realization of a tripartite regular degree sequence D can be transformed into any another realization of D by a series of switch operations.

3. For any odd n, construct an $n \times n$ Latin square containing a transversal of size n.

4. Construct two 5×5 Latin squares such that the first one cannot be transformed into the second one by a series of transformations each affecting only two rows or two columns.

5. Find two 3-uniform hypergraphs H_1 and H_2 with the same degree sequences such that H_1 cannot be transformed into H_2 by a series of switch operations.

6. Find a class of degree sequences $\mathcal{D}$ with the following properties. Any $D = (d_1, d_2, \ldots, d_n) \in \mathcal{D}$ and any 3-uniform realizations H_1 and H_2 of D, H_1 can be transformed into H_2 with a sequence of hinge-flips and switch operations such that any intermediate hypergraph H' has a degree sequence $D' = (d'_1, d'_2, \ldots, d'_n)$ satisfying

$$\sum_i |d_i - d'_i| \leq 2.$$

Note that there are multiple solutions to this problem, try to find as nice solution as possible. Your professor also does not know the ultimate answer to this question, it is part of the proposed research.

References

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