Extremal graph theory, Introduction

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Surveys Introduc1.tex

— My homepage: www.renyi.hu/˜miki
— Erdős homepage: www.renyi.hu/˜p_erdos
— The homepage of Alon, Füredi, . . .

- Alon: Tools from higher algebra, in : "Handbook of Combinatorics", R.L. Graham, M. Grötschel and L. Lovász, eds, North Holland (1995), Chapter 32, pp. 1749-1783.

- Bollobás: Extremal Graph Theory (book)
- Bollobás: B. Bollobás: Extremal graph theory, in: R. L. Graham, M. Grötschel, and L. Lovász (Eds .), Handbook of Combinatorics, Elsevier Science, Amsterdam, 1995, pp. 1231–1292.
- Füredi-Simonovits: The history of degenerate (bipartite) extremal graph problems. Erdős centennial, 169–264, Bolyai Soc. Math. Stud., 25, Budapest, 2013.

Surveys (cont) Introduc1.tex

- Simonovits: Extremal graph problems, Degenerate extremal problems and Supersaturated graphs, Progress in Graph Theory (Acad Press, ed. Bondy and Murty) (1984) 419–437.
- Simonovits: Paul Erdős' influence on extremal graph theory. The mathematics of Paul Erdős, II, 148-192, Algorithms Combin., 14, Springer, Berlin, 1997. (Updated now, 2014 Arxiv)
- M. Simonovits: How to solve a Turán type extremal graph problem? (linear decomposition), Contemporary trends in discrete mathematics (Stirin Castle, 1997), pp. 283–305, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 49, Amer. Math. Soc., Providence, RI, 1999.

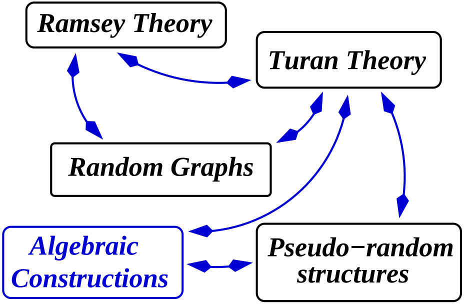
- Keevash
- Kühn-Osthus
- Kohayakawa
- Schacht

These sources were chosen to suit to my lectures, many other very good sources are left out.

Introduction Introduc1.tex

Extremal graph theory and Ramsey theory were among the early and fast developing branches of 20th century graph theory. We shall survey the early development of Extremal Graph Theory, including some sharp theorems.

Strong interactions between these fields: Here everything influenced everything



General Notation Introduc1.tex

- G_n, $Z_{n,k}$, $T_{n,p}$, H_ν... the (first) subscript n will almost always denote the number of vertices.
- K_p = complete graph on p vertices,
- P_k / C_k = path / cycle on k vertices.
- $\delta(x)$ is the degree of the vertex x.
- $v(G)$ / $e(G)$ = # of vertices / edges,
- $\delta(G)$ = mindeg, $\Delta(G)$ = maxdeg
- $\chi(G)$ = the chromatic number of G.
- $N(x)$ = set of neighbours of the vertex x, and
- $G[X]$= the subgraph of G induced by X.
- $e(X, Y)$ = # of edges between X and Y.

Special notation Introduc1.tex

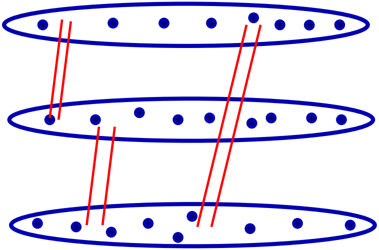
Turán type extremal problems for simple (?) graphs

- Sample graph L, $\mathcal{L}$

$$\mathbf{ex}(n, \mathcal{L}) = \text{ extremal number } = \max_{L \not\subseteq \mathcal{L}} e(G_n).$$

if $L \in \mathcal{L}$

- $\mathbf{EX(n, \mathcal{L})}$ = **extremal graphs**.
- $T_{n,p}$ = **Turán graph**, p-chromatic having most edges.



The Turán Graph

Application in combin. number theory MosTomszk.tex

Erdős (1938): → ErdTomsk
Maximum how many integers $a_i \in [1, n]$ can be found under the condition: $a_i a_j \neq a_k a_\ell$, unless $\{i, j\} = \{k, \ell\}$?

This lead Erdős to prove:

$$\mathbf{ex}(n, C_4) \leq cn\sqrt{n}.$$

The first finite geometric construction to prove the lower bound (Eszter Klein)

Crooks tube

First "attack": MosTomszk.tex

The primes between 1 and n satisfy Erdős' condition.

Can we conjecture $$g(n) \approx \pi(n) \approx \frac{n}{\log n}?$$

YES!

Proof idea: If we can produce each non-prome $m \in [1, n]$ as a product:

$$m = xy, \text{ where } x \in X, \ y \in Y,$$

then

$$g(n) \le \pi(n) + \mathbf{ex}_B(X, Y; C_4).$$

where $\mathbf{ex}_B(U, V; L)$ denotes the maximum number of edges in a subgraph of $G(U, V)$ without containing an L.

The number theoretical Lemma: MosTomszk.tex

Consider only integers. Let $\mathcal{P}$ = primes,

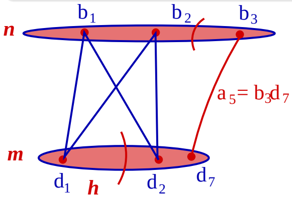
$$\mathcal{B} := [1, n^{2/3}] \bigcup [n^{2/3}, n] \cap \mathcal{P} \text{ and } \mathcal{D} := [1, n^{2/3}].$$

Lemma (Erdős, 1938)

$[1, n] \subseteq \mathcal{B} \cdot \mathcal{D} = (\mathcal{B}_1 \cdot \mathcal{D}) \cup (\mathcal{B}_2 \cdot D)$.

Lemma (Erdős, 1938)

Representing each $a_i = b_i d_i$, the obtained bipartite graph has no C_4.



$e(G(\mathcal{B}_1, \mathcal{D})) \leq 3m\sqrt{m} = 3n$.
$\mathcal{B}_2$ is joined only to $[1, n^{1/3}]$:
$e(G(\mathcal{B}_2, \mathcal{D})) \leq \pi(n) + h^2$
$= \pi(n) + n^{2/3}$.

Kővári – T. Sós – Turán MosDegener1.tex

One of the important extremal graph theorems is that of Kővári, T. Sós and Turán, → KovSosTur

Theorem (Kővári–T. Sós–Turán, → KovSosTur)

Let $K_{a,b}$ denote the complete bipartite graph with a and b vertices in its color-classes. Then

$$\mathbf{ex}(n, K_{a,b}) \le \frac{1}{2}\sqrt[a]{b-1}\cdot n^{2-(1/a)} + O(n).$$

We use this theorem with $a \le b$, since that way we get a better estimate.

Conjecture

The above upper bound is sharp: For every $b \ge a > 0$,

$$\mathbf{ex}(n, K_{a,b}) > c_{a,b} n^{2-(1/a)} + O(n).$$

Is the exponent $2-(1/a)$ sharp? MosDegener1.tex

Conjecture (KST is Sharp)

For every integers a, b,

$$\mathbf{ex}(n, K(a,b)) > c_{a,b}\, n^{2-1/a}.$$

Known for $a=2$ and $a=3$: Finite geometric constructions

Erdős, Rényi, V. T. Sós, → ErdRenyiSos

W. G. Brown → BrownThom

Random methods: → ErdRenyiEvol

$$\mathbf{ex}(n, K(a,b)) > c_a n^{2-\frac{1}{a}-\frac{1}{b}}.$$

Füredi on $K_2(3,3)$: The Brown construction is sharp.

Kollár-Rónyai-Szabó: $b > a!$.

Alon-Rónyai-Szabó: $b > (a-1)!$.

Commutative Algebra constr.

Unknown: MosDegener1.tex

- Missing lower bounds: Constructions needed
- "Random constructions" do not seem to give the right order of magnitude when $\mathcal{L}$ is finite

We do not even know if Pr1

$$\frac{\mathbf{ex}(n, K(4,4))}{n^{5/3}} \to \infty.$$

- Partial reason for the bad behaviour:

Lenz Construction

Problems, Exercises MoszUnitDist.tex

Exercise Let the vertices of a graph be points in $\mathbb{E}^2$ and join two points by an edge if their distance is 1. Show that this graph contains no $K(2,3)$. 2

Exercise Let the vertices of a graph be points in $\mathbb{E}^3$ and join two points by an edge if their distance is 1. Show that this graph contains no $K(3,3)$. 3

Exercise If we take n points of general position in the d-dimensional Euclidean space (i.e., no d of them belong to a $d-1$-dimensional affine subspace) and join two of them if their distance is 1, then the resulting graph G_n can not contain K_{d+2}. 4

Exercise If $a_1, \dots, a_p$ and $b_1, \dots, b_q$ are points in $\mathbb{E}^d$ and all the pairwise distances $\rho(a_i, b_j) = 1$, then the two affine subspaces defined by them are either orthogonal to each other or one of them reduces to one point. 5

Problems, Exercises, cont. MoszUnitDist.tex

Exercise Show that if we join two points in $\mathbb{E}^4$ when their distance is 1, then the resulting graph contains a $K(\infty, \infty)$. 6

Exercise Let $v = v(L)$. Prove that if we put more than $n^{1-(1/v)}$ edges into some class of $T_{n,p}$ then the resulting graph contains L. Can you sharpen this statement? 7

Exercise (Petty's theorem) If we have n points in $\mathbb{E}^d$ with an arbitrary metric $\rho(x, t)$ and its "unit distance graph" contains a K_p then $p \leq 2^d$. (Sharp for the d-dimensional cube and the ℓ_1-metric.) 8*

Erdős on unit distances MoszUnitDist.tex

Many of the problems in the area are connected with the following beautiful and famous conjecture, motivated by some grid constructions.

Conjecture (P. Erdős)

For every $\varepsilon > 0$ there exists an $n_0(\varepsilon)$ such that if $n > n_0(\varepsilon)$ and G_n is the Unit Distance Graph of a set of n points in $\mathbb{E}^2$ then

$$e(G_n) < n^{1+\varepsilon}.$$

The cut lemma BiparLower.tex

Lemma

Erdős triviality Each G_n contains a bipartite subgraph H_n with $e(H_n) > \frac{1}{2}e(G_n)$.

Two proofs. Generalization

Why is the random method weak? BiparLower.tex

Let $\chi(L)=2$, $v:=v(L)$, $e=e(L)$.

- The simple Random method (threshold) gives an L-free graph G_n with $cn^{2-(v/e)}$ edges. For C_{2k} this is too weak.

- Improved method (first moment):

$$cn^{2-\frac{v-2}{e-1}}.$$

For C_{2k} this yields

$$cn^{2-\frac{2k-2}{2k-1}}=cn^{1+\frac{1}{2k-1}}.$$

Conjectured:

$$\mathbf{ex}(n, C_{2k})>cn^{1+\frac{1}{k}}.$$

Random method, General Case: BiparLower.tex

General Lower Bound

If a finite $\mathcal{L}$ does not contain trees (or forests), then for some constants
$c = c_{\mathcal{L}} > 0$, $\alpha = \alpha_{\mathcal{L}} > 0$

$$\mathbf{ex}(n, \mathcal{L}) > cn^{1+\alpha}.$$

Proof (Sketch).

- Discard the non-bipartite L's.

- Fix a large $k = k(\mathcal{L})$. (E.g., $k = \max v(L)$.)
- We know $\mathbf{ex}(n, \{C_4, \dots, C_{2k}\}) > cn^{2-\frac{v-2}{e-1}}$.
- Since each $L \in \mathcal{L}$ contains some $C_{2\ell}$ $(\ell \le k)$,

$$\mathbf{ex}(n, \mathcal{L}) \ge \mathbf{ex}(n, C_4, \dots, C_{2k}) > cn^{1+\frac{1}{2k-1}}.$$

Constructions using finite geometries

$p \approx \sqrt{n} =$ prime ($n = p^2$)
Vertices of the graph F_n are pairs: (a, b) mod p.
Edges: (a, b) is joined to (x, y) if $ac + bx = 1$ mod p.

Geometry in the constructions: the neighbourhood is a straight line and two such nighbourhoods intersect in ≤ 1 vertex.

$\Longrightarrow$ No $C_4 \subseteq F_n$

loops to be deleted
most degrees are around $\sqrt{n}$:

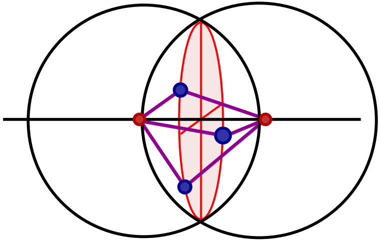
$e(F_n) \approx \frac{1}{2} n\sqrt{n}$

Finite geometries: Brown construction BiparLower.tex

Vertices: (x, y, z) mod p

Edges:

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = \alpha.$$



$$\mathbf{ex}(n, K(3,3)) > \frac{1}{2} n^{2-(1/3)} + o(...).$$

→ BrownThom

The first missing case BiparLower.tex

The above methods do not work for $K(4,4)$.

We do not even know if

Pr2

$$\frac{\mathbf{ex}(n, K_2(4,4))}{\mathbf{ex}(n, K_2(3,3))} \to \infty.$$

One reason for the difficulty: Lenz construction:

$\mathbb{E}^4$ contains two circles in two orthogonal planes:

$$\mathcal{C}_1 = \{x^2+y^2 = \frac{1}{2},\ z=0,\ w=0\} \text{ and } \mathcal{C}_2 = \{z^2+w^2 = \frac{1}{2},\ x=0,\ y=0\}$$

and each point of $\mathcal{C}_1$ has distance 1 from each point of $\mathcal{C}_2$: the unit distance graph contains a $K_2(\infty,\infty)$.

Theorem (Erdős–Simonovits, Cube Theorem)

Let Q_8 denote the cube graph defined by the vertices and edges of a 3-dimensional cube. Then

$$\mathbf{ex}(n, Q_8) = O(n^{8/5}).$$

Exponents? MosDegenerate2.tex

Conjecture (Erdős and Simonovits, Rational exponents)

For any finite family $\mathcal{L}$ of graphs, if there is a bipartite $L \in \mathcal{L}$, then there exist a rational $\alpha \in [0,1)$ and a $c > 0$ such that

$$\frac{\mathbf{ex}(n, \mathcal{L})}{n^{1+\alpha}} \to c.$$

Classification of extremal graph problems and lower bound constructions MoszkvaF1a.tex

- The asymptotic structure of extremal graphs
- Degenerate extremal graph problems:
  – $\mathcal{L}$ contains a bipartite L:
  – $\mathbf{ex}(n, \mathcal{L}) = o(n^2)$.
- Lower bounds using random graphs and finite geometries:
  – Here random methods are weak
  – Finite geometry sometimes gives sharp results.

The Erdős-Stone theorem (1946) MoszkvaF1a.tex

$$\mathbf{ex}(n, K_{p+1}(t,\dots,t)) = \mathbf{ex}(n, K_{p+1}) + o(n^2)$$

Motivation from topology

General asymptotics MoszkvaF1a.tex

Erdős-Stone-Sim.

→ ErdSimLim

If

$$\min_{L\in\mathcal{L}} \chi(L) = p + 1$$

then

$$\mathbf{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

So the asymptotics depends only on the

minimum chromatic number

Erdős-Stone-Sim. thm MoszkvaF1a.tex

$$\mathbf{ex}(n, \mathcal{L}) = \mathbf{ex}(n, K_{p+1}) + o(n^2).$$

How to prove this from ERDŐS-STONE?

- pick $L \in \mathcal{L}$ with $\chi(L) = p + 1$.
- pick t with $L \subseteq K_{p+1}(t, \dots, t)$.
- apply ERDŐS-STONE:

$$\mathbf{ex}(n, \mathcal{L}) \ge e(T_{n,p})$$

but

$$\mathbf{ex}(n, \mathcal{L}) \le \mathbf{ex}(n, L) \le \mathbf{ex}(n, K_{p+1}(t, \dots, t)) \le e(T_{n,p}) + \varepsilon n^2.$$

Classification of extremal problems

- nondegenerate: $p > 1$
- degenerate: $\mathcal{L}$ contains a bipartite L
- strongly degenerate: $T_\nu \in \mathcal{M}(\mathcal{L})$

where $\mathcal{M}$ is the decomposition family.

Main Line: MoszkvaF1a.tex

Many central theorems

assert that for ordinary graphs the general situation is almost the same as for K_{p+1}.

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

- The extremal graphs S_n are very similar to $T_{n,p}$.
- the almost extremal graphs are also very similar to $T_{n,p}$.

The meaning of "VERY SIMILAR": MoszkvaF1a.tex

- One can delete and add $o(n^2)$ edges of an extremal graph S_n to get a $T_{n,p}$.
- One can delete $o(n^2)$ edges of an extremal graph to get a p-chromatic graph.

Stability of the class sizes MoszkvaF1a.tex

Exercise Among all the n-vertex p-chromatic graphs $T_{n,p}$ is the (only) graph maximizing $e(T_{n,p})$. Tp1

Exercise (Stability) If $\chi(G_n) = p$ and

$$e(G_n) = e(T_{n,p}) - s$$

then in a p-colouring of G_n, the size of the i^{th} colour-class,

$$\left| n_i - \frac{n}{p} \right| < c\sqrt{s+1}.$$

Tp2

Exercise Prove that if n_i is the size of the i^{th} class of $T_{n,p}$ and G_n is p-chromatic with class sizes $m_1, \dots, m_p$, and if $s_i := |n_i - m_i|$, then

$$e(G_n) \le e(T_{n,p}) - \sum \binom{s_i}{2}.$$

Prove the assertion of the previous exercise from this. Tp2b

Extremal graphs MoszkvaF1a.tex

The "metric" $\rho(G_n, H_n)$ is the minimum number of edges to change to get from G_n a graph isomorphic to H_n.

Notation.

$\mathbf{EX}(\mathbf{n}, \mathcal{L})$: set of extremal graphs for $\mathcal{L}$.

Theorem (Erdős-Sim., 1966)

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

If $S_n \in \mathbf{EX}(\mathbf{n}, \mathcal{L})$, *then*

$$\rho(T_{n,p}, S_n) = o(n^2).$$

Product conjecture MoszkvaF1a.tex

Theorem 1 separates the cases $p = 1$ and $p > 1$:

$$\mathbf{ex}(n, \mathcal{L}) = o(n^2) \iff p = p(\mathcal{L}) = 1$$

$p = 1$: degenerate extremal graph problems

Conjecture (Sim.)

If

$$\mathbf{ex}(n, \mathcal{L}) > e(T_{n,p}) + n \log n$$

and $S_n \in \mathbf{EX(n}, \mathcal{L})$, *then* S_n *can be obtained from a* $K_p(n_1, \dots, n_p)$ *only by adding edges.*

This would reduce the general case to degenerate extremal graph problems:

The product conjecture, Reduction

Definition

Given the vertex-disjoint graphs $H_1, \dots, H_p$, their product $\prod_{i=1}^{p} H_{n_i}$ is the graph H_n obtained by joining all the vertices of H_{n_i} to all vertices of H_{n_j}, for all $1 \le i < j \le p$.

Exercise Prove that if $\prod_{i=1}^{p} H_{n_i}$ is extremal for $\mathcal{L}$ then H_{n_1} is extremal for some $\mathcal{M}_1$. (Hint: Prove this first for $p = 1$.) Redu

Definition (Decomposition)

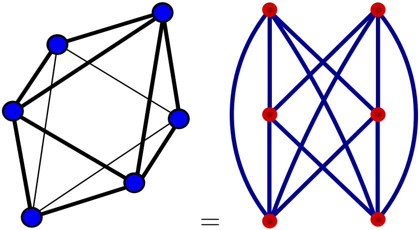
M is a decomposition graph for $\mathcal{L}$ if some $L \in \mathcal{L}$ can be $p+1$-colored so that the first two colors span an M^* containing M. $\mathcal{M} = \mathcal{M}(\mathcal{L})$ is the family of decomposition graphs of $\mathcal{L}$.

Exercise Prove that if $\prod_{i=1}^{p} H_{n_i}$ is extremal for $\mathcal{L}$ then H_{n_i} is extremal for some $\mathcal{M}_i \subseteq \mathcal{M}$ and $p(\mathcal{M}) = 1$: the problem of $\mathcal{M}$ is **degenerate**. Redu2

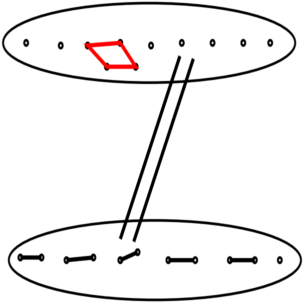
Example: Octahedron Theorem MoszkvaF1a.tex

Theorem (Erdős-Sim.)

For O_6, the extremal graphs S_n are "products": $U_m \otimes W_{n-m}$ where U_m is extremal for C_4 and W_{n-m} is extremal for P_3. for $n > n_0$. → ErdSimOcta



EXCLUDED: OCTAHEDRON

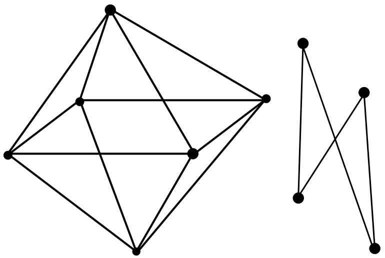


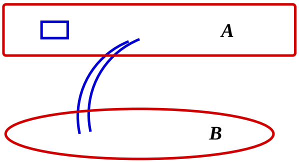
EXTREMAL = PRODUCT

Decomposition decides the error terms MoszkvaF1a.tex

Definition (Decomposition, alternative def.)

For a given $\mathcal{L}$, $\mathcal{M} := \mathcal{M}(\mathcal{L})$, $\mathcal{M}$ is the family of all those graphs M for which there is an $L \in \mathcal{L}$ and a $t = t(L)$ such that $L \subseteq M \otimes K_{p-1}(t, \dots, t)$.
We call $\mathcal{M}$ the **decomposition family** *of $\mathcal{L}$.*





If B contains a C_4 then G_n contains an octahedron: K(3,3,3).

The product conjecture, II. MoszkvaF1a.tex

Conjecture (Product)

If no p-chromatic $L \in \mathcal{L}$ can be $p+1$-colored so that the first two color classes span a tree (or a forest) then all (or at least one of) the extremal graphs are products of p subgraphs of size $\approx \frac{n}{p}$.

Structural stability MoszkvaF1b.tex

Erdős-Sim. Theorem.

Put

$$p := \min_{L\in\mathcal{L}} \chi(L) - 1.$$

For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $L \not\subseteq G_n$ for any $L \in \mathcal{L}$ and

$$e(G_n) \ge \left(1 - \frac{1}{p}\right) \binom{n}{2} - \delta n^2,$$

then

$$\rho(G_n, T_{n,p}) \le \varepsilon n^2$$

Structural stability: $o(.)$ form MoszkvaF1b.tex

Erdős-Sim. Theorem

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

If G_n is almost extremal:

- It is $\mathcal{L}$-free, and
- $$e(G_n) \geq \left(1 - \frac{1}{p}\right) \binom{n}{2} - o(n^2),$$

then

$$\rho(G_n, T_{n,p}) = o(n^2).$$

Corollary

The almost extremal graphs are almost-p-colorable

Improved error terms, depending on $\mathcal{M}$.

Erdős-Sim. Theorem.

Put

$$p := \min_{L\in\mathcal{L}} \chi(L) - 1.$$

Let $\mathcal{M} = \mathcal{M}(\mathcal{L})$ be the decomposition family. Let $\mathbf{ex}(n, \mathcal{M}) = O(n^{2-\gamma})$. Then, if G_n is almost extremal:

- It is $\mathcal{L}$-free, and
- $$e(G_n) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - O(n^{2-\gamma}),$$

then we can delete $O(n^{2-\gamma})$ edges from G_n to get a p-chromatic graph.

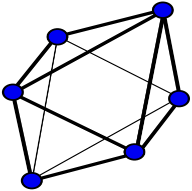
Remark

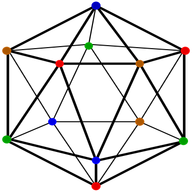
For extremal graphs $\rho(S_n, T_{n,p}) = O(n^{2-\gamma})$.

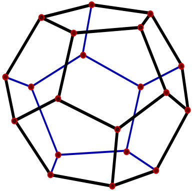
Applicable and gives also exact results MoszkvaF1b.tex

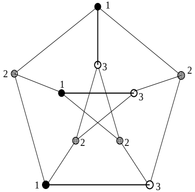
Examples:

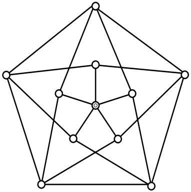
Octahedron, Icosahedron, Dodecahedron, Petersen graph, Grötzsch







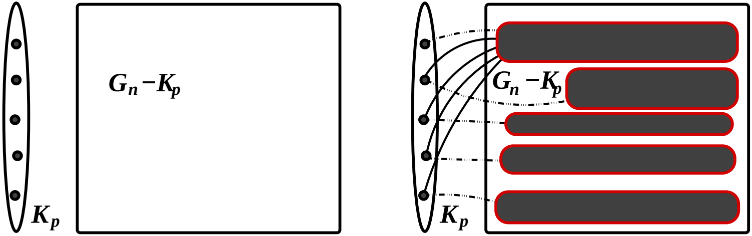




In all these cases the stability theorem yields exact structure for $n > n_0$.

Original proof of Turán's thm MoszkvaF1b.tex

- We may assume that $K_p \subseteq G_n$.
- We cut off K_p.
- We use induction on n (from $n - p$).

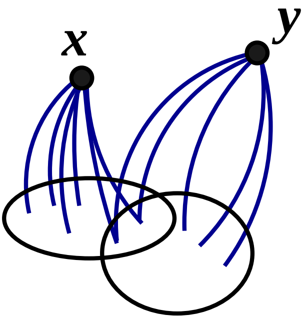


- We show the uniqueness

This "splitting off" method can be used to prove the structural stability and many other results. However, there we split of, say a large but fixed $K_p(M, \ldots, M)$.

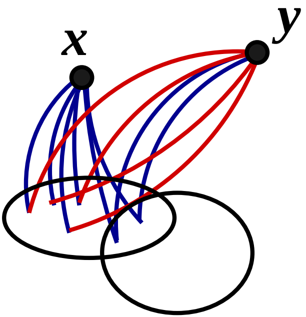
Zykov's proof, 1949 MoszZykovProof.tex

...and why do we like it?



Assume $deg(x) \geq deg(y)$.

Zykov's proof, 1949. MoszZykovProof.tex



Lemma. If $G_n \not\supseteq K_\ell$ and we symmetrize, the resulting graph will neither contain a K_ℓ.

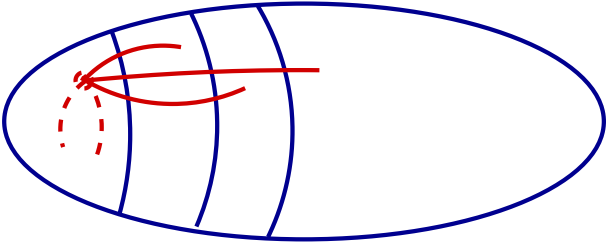
We replace $N(x)$ by $N(y)$.

- Algorithmic proof
- Applicable in many cases
- Equivalent with Motzkin-Straus

How to use this? MoszZykovProof.tex

We can use a parallel symmetrization.

• = max degree



Uniqueness?

- Füredi proved the stability for K_{p+1}, analyzing this proof:

If there are many edges among the nonneighbours of the base x_i then we gain a lot.

Other directions MoszkvaF1c.tex

- Prove exact results for special cases
- Prove good estimates for the bipartite case: $p = 1$
- Clarify the situation for digraphs
- Prove reasonable results for hypergraphs

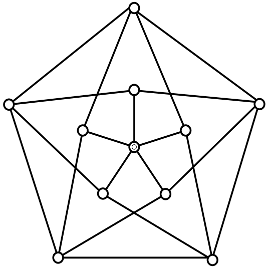
- Investigate fields where the problems have other forms, yet they are strongly related to these results.

Examples: 1. Critical edge MoszkvaF1c.tex

Theorem (Critical edge)

If $\chi(L) = p + 1$ and L contains a color-critical edge, then $T_{n,p}$ is the (only) extremal for L, for $n > n_1$. *[If and only if]*

Sim., (Erdős)



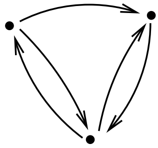
Grötzsch graph

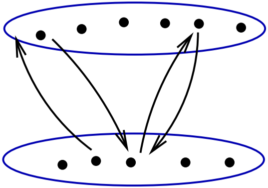
Complete graphs
Odd cycles

Examples: 2. A digraph theorem MoszkvaF1c.tex

We have to assume an upper bound s on the multiplicity. (Otherwise we may have arbitrary many edges without having a K_3.) Let $s = 1$.

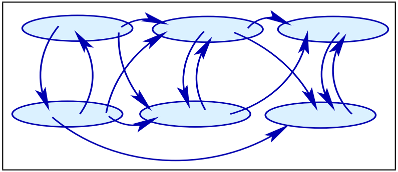
L:





$$\mathbf{ex}(n, L) = 2\mathbf{ex}(n, K_3) \qquad (n > n_0?)$$

Many extremal graphs: We can combine arbitrary many oriented double Turán graph by joining them by single arcs.

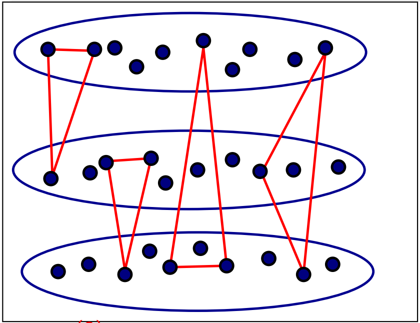


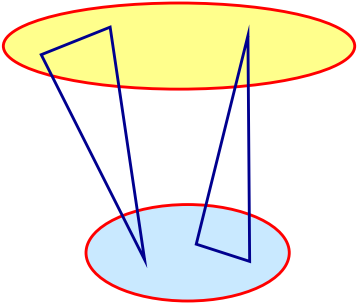
Example 3. The famous Turán conjecture

Consider 3-uniform hypergraphs.

Conjecture (Turán)

The following structure (on the left) is the (? asymptotically) extremal structure for $K_4^{(3)}$*:*





For $K_5^{(3)}$ one conjectured extremal graph is just the above

Examples: Degree Majorization MoszkvaF1c.tex

Erdős

For every K_{p+1}-free G_n there is a p-chromatic H_n with

$$d_H(v_i) \geq d_G(v_i).$$

(I.e the degrees in the new graph are at least as large as originally.)

Bollobás-Thomason, Erdős-T. Sós

If $e(G_n) > e(T_{n,p})$ edges, then G_n has a vertex v with

$$e(G[N(v)]) \geq \mathbf{ex}(d(v), K_p).$$

(I.e the neighbourhood has enough edges to ensure a K_p.)
Both generalize the Turán thm.

Application of symmetrization MoszkvaF1c.tex

Exercise Prove that symetrization does not produce new complete graphs: if the original graph did not contain K_ℓ, the new one will neither. NN

Exercise Prove the degree-majorization theorem, using symmetrization. EM

Exercise (Bondy) Prove the Bollobás-Thomason- Erdős-T. Sós theorem, using symmetrization. Bo

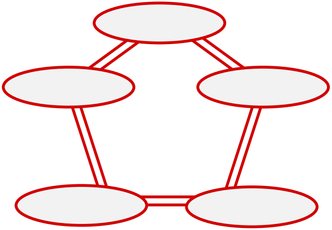
Exercise Is it true that if a graph does not contain C_4 and you symmetrize, the new graph will neither contain a C_4? Sy

Examples: MoszkvaF1c.tex

Prove that each triangle-free graph can be turned into a bipartite one deleting at most $n^2/25$ edges.

The construction shows that this is sharp if true.

Partial results: Erdős-Faudree-Pach-Spencer



Erdős-Győri-Sim.
Győri
Füredi

Erdős-Sós conjecture MoszkvaErdSosTrees.tex

$$\mathbf{ex}(n, T_k) \le \frac{1}{2}(k-1)n.$$

AJTAI-KOMLÓS-SIM.-SZEMERÉDI: True if $k > k_0$.

Importance of Decomposition MoszDecompZ.tex

This determines the real error terms in our theorems. E.g., if $\mathcal{M}$ is the family of decomposition graphs.

$$e(T_{n,p}) + \mathbf{ex}(n/p, \mathcal{M}) \leq \mathbf{ex}(n, \mathcal{L}) \leq e(T_{n,p}) + c \cdot \mathbf{ex}(n/p, \mathcal{M})$$

for any $c > p$, and n large.

Exercise What is the decomposition class of K_{p+1}? D1

Exercise What is the decomposition class of the octahedron graph $K_3(2,2,2)$? More generally, of $K(p,q,r)$? D2

Exercise What is the decomposition class of the Dodecahedron graph D_{20}? And of the icosahedron graph I_{12}? D1

The corresponding theorems MoszDecompZ.tex

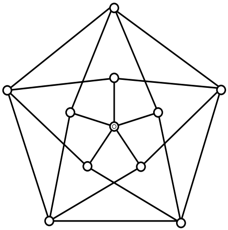
Definition

e is color-critical edge if $\chi(L - e) < \chi(L)$.

Theorem (Critical edge, (Sim.))

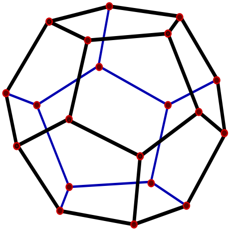
If $\chi(L) = p + 1$ *and* L *contains a color-critical edge, then* $T_{n,p}$ *is the (only) extremal for* L*, for* $n > n_1$.

+ Erdős

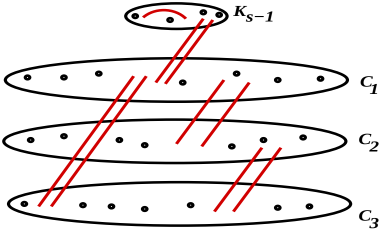


Complete graphs
Odd cycles

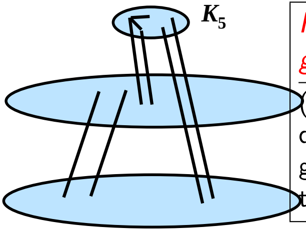
Dodecahedron Theorem (Sim.) MoszDecompZ.tex



Dodecahedron: D_{20}



$H(n, d, s)$

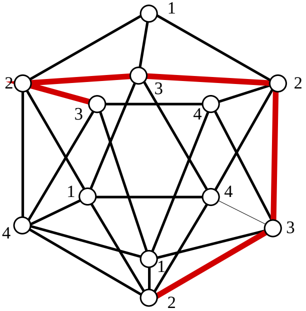


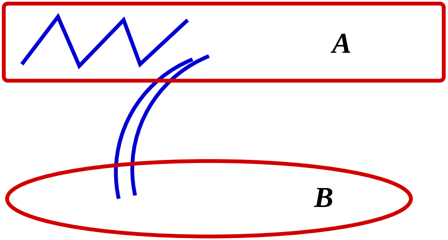
$H(n, 2, 6)$

For D_{20}, $H(n, 2, 6)$ is the (only) extremal graph for $n > n_0$.

($H(n, 2, 6)$ cannot contain a D_{20} since one can delete 5 points of $H(n, 2, 6)$ to get a bipartite graph but one cannot delete 5 points from D_{20} to make it bipartite.)

Example 2: the Icosahedron MoszDecompZ.tex





If B contains a P_6, then G_n contains an icosahedron

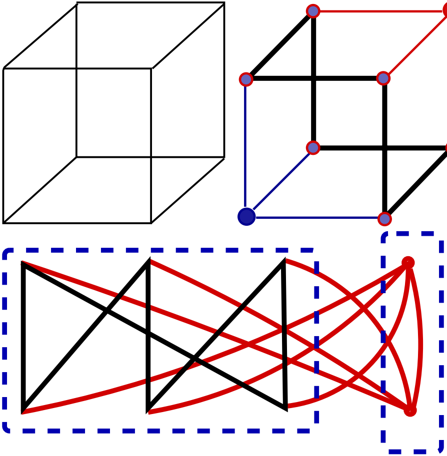
The decomposition class is: P_6.

Cube-reduction MoszkvaF1d.tex

Theorem (Cube, Erdős-Sim.)

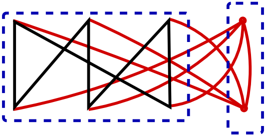
$\mathbf{ex}(n, Q_3) = O(n^{8/5})$.

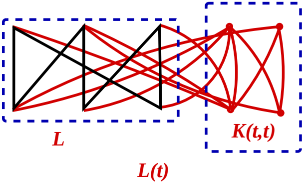
New Proofs: PINCHASI-SHARIR, FÜREDI, ...



General definition of $L(t)$: MoszkvaF1d.tex

- Take an arbitrary bipartite graph L and $K(t,t)$. 2-color them!
- join each vertex of $K(t,t)$ to each vertex of L of the opposite color





Theorem (Reduction, Erdős-Sim.)

Fix a bipartite L and an integer t.
If $\mathbf{ex}(n,L) = n^{2-\alpha}$ *and* $L(t)$ *is defined as above then* $\mathbf{ex}(n,L(t)) \le n^{2-\beta}$ *for*

$$\frac{1}{\beta} - \frac{1}{\alpha} = t.$$

Examples MoszkvaF1d.tex

The ES reduction included many (most?) of the earlier upper bounds on bipartite L. Deleting an edge e of L, denote by $L - e$ the resulting graph.

Exercise Deduce the KST theorem from the Reduction Theorem. A

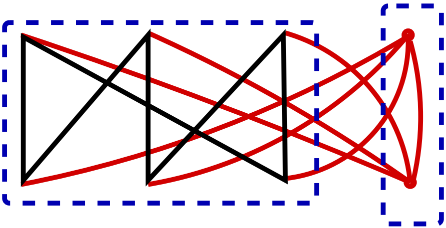
Exercise Show that $\mathbf{ex}(n, Q_8 - e) = O(n^{3/2})$. B

Exercise Show that $\mathbf{ex}(n, K_2(p,p) - e) = O(n^{2-(1/p)})$. C

Open Problem: Pr3

Find a lower bound for $\mathbf{ex}(n, Q_8)$, better than $cn^{3/2}$.
Conjectured: $\mathbf{ex}(n, Q_8) > cn^{8/5}$.

How to get $\mathbf{ex}(n, Q_8) = O(n^{8/5})$? MoszkvaF1d.tex



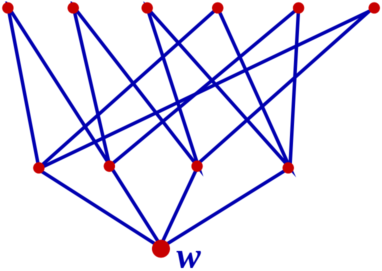
$Q_8 = C_6(1)$. Apply
$\mathbf{ex}(n, C_6) = O(n^{2-(2/3)})$ with $\alpha = 2/3$,
$t = 1$
Use the Reduction Thm:

$$\frac{1}{\beta} - \frac{1}{\alpha} = t.$$

Now $\frac{1}{\beta} - \frac{1}{\alpha} = 1$. So $\frac{1}{\beta} = 1 + \frac{3}{2} = \frac{5}{2}$. Hence $2 - \beta = 2 - \frac{2}{5} = \frac{8}{5}$

What is left out? MoszkvaF1d.tex

The graph F_{11} below is full of C_4's.



Erdős conjectured that $\mathbf{ex}(n, F_{11}) = O(n^{3/2})$. The methods known tose days did not give this. Füredi proved the conjecture. → Fur11CCA

The general definition: In $F_{1+k+\binom{k}{\ell}}$ w is joined to k vertices $x_1, \dots, x_k$, and $\binom{k}{\ell}$ further vertices are joined to each ℓ-tuple $x_{i_1} \dots x_{i_\ell}$.
$F_{11} = F_{1+4+\binom{4}{2}}$.

Bondy-Simonovits MoszBondySim.tex

Theorem (Even Cycle: C_{2k})

$\mathbf{ex}(n, C_{2k}) = O(n^{1+(1/k)})$.

More explicitly:

Theorem

Even Cycle: C_{2k}). $\mathbf{ex}(n, C_{2k}) \le c_1 k n^{1+(1/k)}$.

Conjecture (Sharpness)

Is this sharp, at least in the exponent? The simplest unknown case is C_8,

It is sharp for C_4, C_6, C_{10}

Could you reduce k in $c_1 k n^{1+(1/k)}$?

Sketch of the proof: MoszBondySim.tex

Lemma

If D is the average degree in G_n, then G_n contains a subgraph G_m with

$$d_{\min}(G_m) \geq \frac{1}{2}D \text{ and } m \geq \frac{1}{2}D.$$

Exercise Can you improve this lemma? Im

- So we may assume that G_n is bipartite and regular.

Assume also that it does not contain shorter cycles either.

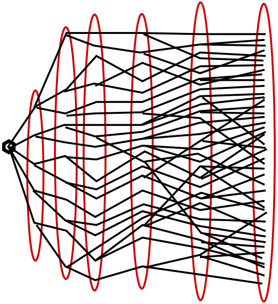
Sketch of the proof: Expansion MoszBondySim.tex

Start with cheating: girth $> 2k$:

- The ith level contains at least D^i different points.
- $D^i < n$, $i = 1, 2, \dots k$.

So $D < n^{1/k}$.

- $e(G_n) \le cDn \le \frac{1}{2} n^{1+1/k}$.



We still have the difficulty that the shorter cycles cannot be trivially eliminate methods to overcome this:

- Bondy-Simonovits and → BondySim
- Faudree-Simonovits → Fau

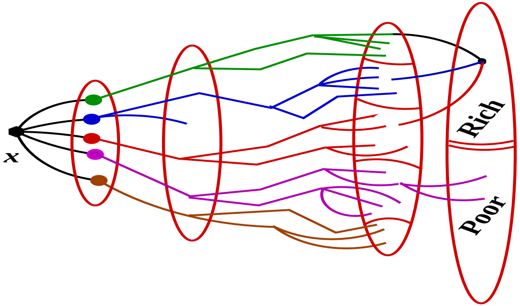
Both proofs use Expansion: MoszBondySim.tex

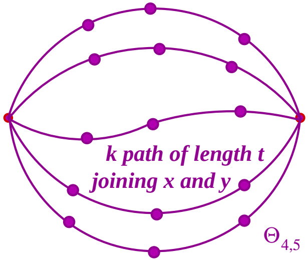
x is a fixed vertex, S_i is the $i^{\rm th}$ level, we need that

$$\frac{|S_{i+1}|}{|S_i|} > c_L \cdot d_{\min}(G_n) \text{ for } i = 1, \dots, k.$$

Faudree-Simonovits method: MoszBondySim.tex

This gives more: $\mathbf{ex}(n, \Theta_{k,\ell}) = O(n^{1+(1/k)})$.





An Erdős problem: Compactness? MoszkvaF1.tex

We know that if G_n is bipartite, C_4-free, then

$$e(G_n) \le \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

We have seen that there are C_4-free graphs G_n with

$$e(G_n) \approx \frac{1}{2} n^{3/2} + o(n^{3/2}).$$

Conjecture (Erdős Pr4)

Is it true that if $K_3, C_4 \not\subseteq G_n$ then

$$e(G_n) \le \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}) \ ?$$

This does not hold for hypergraphs (Balogh) or for geometric graphs (Tardos)

Erdős-Sim., C_5-compactness: MoszkvaF1.tex

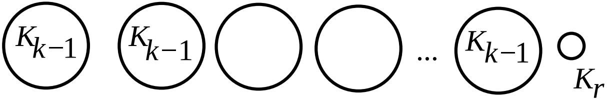
If $C_5, C_4 \not\subseteq G_n$ then → ErdSimComp

$$e(G_n) \leq \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

Unfortunately, this is much weaker than the conjecture on C_3, C_4: excluding a C_5 is a much more restrictive condition.

Erdős-Gallai: MoszkvaF1.tex

$$\mathbf{ex}(n, P_k) \le \frac{1}{2}(k-2)n.$$



Faudree-Schelp
Kopylov

Erdős-T. Sós: MoszkvaF1.tex

Conjecture (Extremal number of the trees)

For any tree T_k,

$$\mathbf{ex}(n, T_k) \le \tfrac{1}{2}(k-2)n.$$

- *Motivation: True for the two extreme cases: path and star.*
- *fight for* $\frac{1}{2}$
- *Partial results*

Theorem (Andrew McLennan)

The Erdős-Sós conjecture holds for trees of diameter 4, *(2003)*

Theorem (Ajtai-Komlós-Sim.-Szemerédi)

If $k > k_0$ *then true:*

$$\mathbf{ex}(n, T_k) \le \frac{1}{2}(k-2)n.$$

Which type of methods? MoszkvaF1.tex

- Stability Method
- Double counting, Cauchy-Schwartz
  — Lovász-Szegedy, Hatami-Norine
- Random Graphs
- Finite Geometries:
  — Klein, Reiman, Erdős-Rényi-Sós
- Erdős: $\mathbf{ex}(n, C_3, \ldots, C_{2k}) > cn^{1+\frac{1}{k}}$
- Eigenvalue questions / technique
  — Guiduli, Babai, Nikiforov ... and many others?
- Szemerédi Regularity Lemma
- Quasi-randomness
  — Simonovits-Sós
- Generalized quasi-randomness, Lovász-Sós-...

Lower bounds for degenerate cases MoszkvaF1.tex

- Why is the random method weak?
- Why is the Lenz construction important?
- Finite geometries
- Commutative algebra method
- Kollár-Rónyai-Szabó
- Alon-Rónyai-Szabó
- Margulis-Lubotzky-Phillips-Sarnak method

→ Margu
→ LubPhilSar

- Lazebnik-Ustimenko-Woldar
- Even cycle-extremal graphs

Rational exponents? MoszkvaF1.tex

Conjecture (Rational exponents, Erdős-Sim.)

Given a bipartite graph L, is it true that for suitable $\alpha \in [0,1)$ there is a $c_L > 0$ for which

$$\frac{\mathbf{ex}(n,L)}{n^{1+\alpha}} \to c_L > 0 \quad ?$$

Or, at least, is it true that for suitable $\alpha \in [0,1)$ there exist a $c_L > 0$ and a $c_L^* > 0$ for which

$$c_1^* \le \frac{\mathbf{ex}(n,L)}{n^{1+\alpha}} \le c_L \quad ?$$

The Universe MoszkvaF1.tex

Extremal problems can be asked (and are asked) for many other object types.

- Mostly simple graphs
- Digraphs → Brown-Harary, Brown, Erdős, Simonovits
- Multigraphs → Brown-Harary, Brown, Erdős, Simonovits
- Hypergraphs → Turán, . . .
- Geometric graph → Pach, Tóth, Tardos
- Integers → Erdős,Sidon,Szemerédi,. . .
- groups
- other structures

The general problem MoszkvaF1.tex

Given a **universe**, and a structure $\mathbb{A}$ with two (natural parameters) n and e on its objects G.
Given a property $\mathcal{P}$.

$$\mathbf{ex}(n,\mathcal{P}) = \max_{n(G)=n} e(G).$$

Determine $\mathbf{ex}(n,\mathcal{P})$ and
describe the **EXTREMAL STRUCTURES**

Examples: Hypergraphs, . . . MoszkvaF1.tex

We return to this later.

Examples: Multigraphs, Digraphs, . . . MoszkvaF1.tex

-
- BROWN-HARARY: bounded multiplicity: r
- BROWN-ERDŐS-SIM. → BrownSimDM

$r = 2s$: digraph problems and multigraph problems seem to be equivalent:

– each multigraph problem can easily be reduced to digraph problems

– and we do not know digraph problems that are really more difficult than some corresponding multigraph problem

Examples: Numbers, . . . MoszkvaF1.tex

- Tomsk
- Sidon sequences

- Let $r_k(n)$ denote the maximum m such that there are m integers $a_1, \dots, a_m \in [1, n]$ without k-term arithmetic progression.

Theorem (Szemerédi Theorem)

For any fixed k $r_k(n) = o(n)$ as $n \to \infty$.

History (simplified):

- K. F. Roth: $r_3(n) = o(n)$
- Szemerédi
- Fürstenberg: Ergodic proof
- Fürstenberg-Katznelson: Higher dimension
- Polynomial extension, Hales-Jewett extension
- Gowers: much more effective

Extremal hypergraph graph theory,

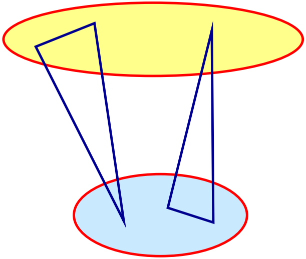
Miklos Simonovits

Moscow, 2015

Hypergraph extremal problems MoszkHypergr.tex

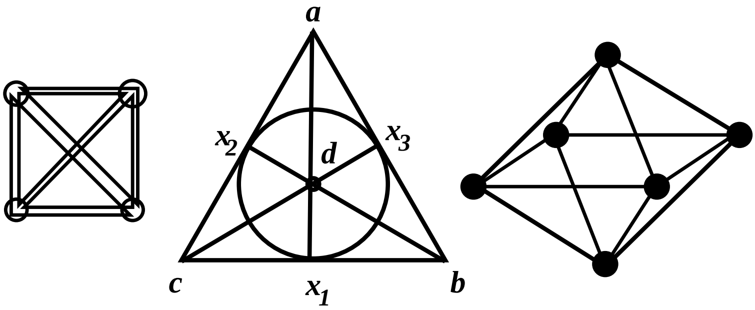
3-uniform hypergraphs: $\mathcal{H} = (V, \mathcal{H})$
$\chi(\mathcal{H})$: the minimum number of colors needed to have in each triple 2 or 3 colors.

Bipartite 3-uniform hypergraphs:



The edges intersect both classes

Three important hypergraph cases MoszkHypergr.tex

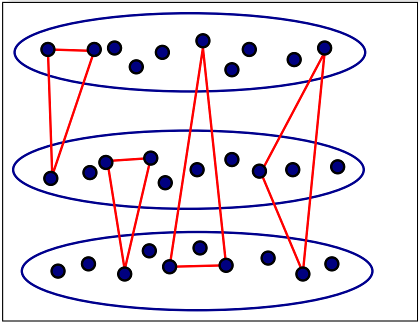


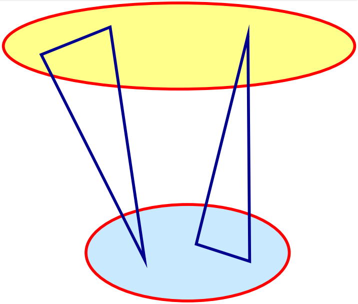
Complete 4-graph, || Fano configuration, || octahedron graph

The famous Turán conjecture MoszkHypergr.tex

Conjecture (Turán)

The following structure is the (? asymptotically) extremal structure for $K_4^{(3)}$*:*



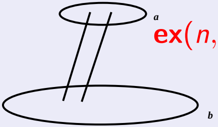


For $K_5^{(3)}$ one conjectured extremal graph is just the above "complete bipartite" one!

Two important theorems MoszkHypergr.tex

Theorem (Kővári-T. Sós-Turán)

Let $2 \le a \le b$ be fixed integers. Then

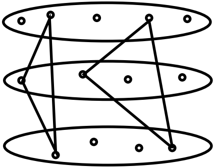


$$\mathbf{ex}(n, K(a,b)) \le \frac{1}{2}\sqrt[a]{b-1}\,n^{2-\frac{1}{a}} + \frac{1}{2}an.$$

→ *KovSosTur*

Theorem (Erdős)

$$\mathbf{ex}(n, K_r^{(r)}(m,\dots,m)) = O(n^{r-(1/m^{r-1})}).$$



How to apply this? MoszkHypergr.tex

Call a hypergraph extremal problem (for k-uniform hypergraphs) degenerate if

$$\mathbf{ex}(n, \mathcal{L}) = o(n^k).$$

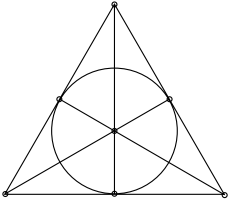
Degenerate hypergraph problems MoszkHypergr.tex

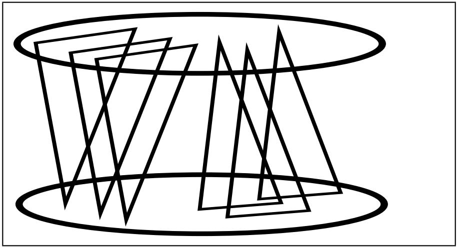
Exercise Prove that the problem of L is degenerate iff it can be k-colored so at each edge meats each of the k colors. XX

The T. Sós conjecture MoszkHypergr.tex

Conjecture (V. T. Sós)

Partition $n > n_0$ vertices into two classes A and B with $||A| - |B|| \leq 1$ and take all the triples intersecting both A and B. The obtained 3-uniform hypergraph is extremal for $\mathcal{F}$.





The conjectured extremal graphs: $\mathcal{B}(X, \overline{X})$

Füredi-Kündgen Theorem MoszkHypergr.tex

If M_n is an arbitrary multigraph (without restriction on the edge multiplicities, except that they are nonnegative) and all the 4-vertex subgraphs of M_n have at most 20 edges, then

$$e(M_m) \leq 3\binom{n}{2} + O(n).$$

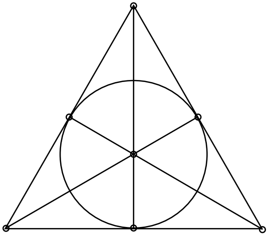
→ FureKund

Theorem (de Caen and Füredi)

→ *FureCaen*

$$\mathbf{ex}(n, \mathcal{F}) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

The Fano-extremal graphs MoszkHypergr.tex



Main theorem. *If $\mathcal{H}$ is a triple system on $n > n_1$ vertices not containing $\mathcal{F}$ and of maximum cardinality, then $\chi(\mathcal{H}) = 2$.*

$$\Longrightarrow \qquad \mathrm{ex}_3(n, \mathcal{F}) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

Remark

The same result was proved independently, in a fairly similar way, by

Peter Keevash and Benny Sudakov → KeeSud .

Theorem (Stability)

There exist a $\gamma_2 > 0$ and an n_2 such that:
If $\mathcal{F} \not\subseteq \mathcal{H}$ and

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2} \text{ for each } x \in V(\mathcal{H}),$$

then $\mathcal{H}$ is bipartite, $\mathcal{H} \subseteq \mathcal{H}(X, \overline{X})$. → *FureSimFano*

Many thanks for your attention.