HOMEWORK SET 4

9) The goal of this problem is to prove the crucial $L(1,\chi) \neq 0$ in an alternative way. Let $q \in \mathbf{N}$, and the characters below are the characters modulo q, χ_0 standing for the principal character, Ξ is the set of all such characters.

   a) *(0 point)* Prove (in an alternative way to what we do in the lecture) that if $\chi \neq \chi_0$, then $L(s,\chi)$ is holomorphic on the domain $\Re s > 0$. (*Hint:* recall that $\sum_{n=1}^{\infty} \chi(n) n^{-s}$ is convergent for $s > 0$ from the lecture (Proposition 1.4.1 and its neighborhood), and combine this with Homework problem 7.)

   b) *(1 point)* Consider the product
   $$F(s) = \prod_{\chi \in \Xi} L(s,\chi),$$
   and prove that, going for contradiction, if any of $L(1,\chi)$ vanishes, this extends holomorphically to $\Re s > 0$.

   c) *(4 points)* For $\Re s > 1$, $F(s)$ is a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$. Prove that the coefficients are nonnegative real numbers (i.e. $a_n \geqslant 0$). (*Hint:* use the Euler product of each $L(s,\chi)$, and show that for each prime p not dividing q, the product (over $\chi \in \Xi$) of Euler factors at p is $(1 - p^{-\alpha_p s})^{-\beta_p}$ for some positive integers α_p, β_p depending only on p (and the all along implicit q).)

   d) *(0 point)* Prove that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is convergent on $\Re s > 0$. (*Hint:* use Homework problem 8.)

   e) *(4 points)* Conclude contradiction by showing that the Dirichlet series cannot converge at $s = 1/\varphi(q)$. (*Hint:* understand what α_p, β_p from above are, and recall that the reciprocal sum of primes is infinity.)

10) *(3 points)* For some $0 < \delta \leqslant 1/2$, assume that $\zeta(s) \neq 0$, if $\Re s > 1 - \delta$ (this is the Riemann Hypothesis with $\delta = 1/2$). Prove that under this condition,
$$\sum_{n \leqslant x} \Lambda(n) = x + O(x^{1-\delta} \log^2 x).$$
(*Hint:* apply Proposition 2.5.1. Be careful with potential zeros on the line $\Re s = 1 - \delta$, try to figure out how to treat them.)