HOMEWORK SET 3

7) *(8 points)* Prove that if a Dirichlet series is convergent at a certain point $s_0 \in \mathbf{C}$, then it is locally uniformly convergent on the half-plane $\{s \in \mathbf{C} : \Re s > \Re s_0\}$. (*Hint:* let $\sum_{n=1}^{\infty} a_n n^{-s}$ be the Dirichlet series, and let C be the compact set, $\varepsilon > 0$ the small number in the input of local uniform convergence. Include C in the interior of an angular domain $|\Im s - \Im s_0|/(\Re s - \Re s_0) < H$ for some fixed $H \in \mathbf{R}$. Define $R(u) = \sum_{n>u} a_n n^{-s_0}$ to be the tail at s_0. For any integers $M < N$, express $\sum_{n=M+1}^{N} a_n n^{-s}$ in terms of $R(M), R(N)$ and $R(u)$ for $M \leqslant u < N$, using partial summation. Prove that $|n^{s_0-s} - (n-1)^{s_0-s}| \leqslant |s_0 - s| \int_{n-1}^{n} w^{\Re s_0 - \Re s - 1} dw$. Choose M such that $|R(u)| < \varepsilon$ for $u \geqslant M$, and put the pieces together to get an estimate on $\sum_{n=M+1}^{N} a_n n^{-s}$ in terms of H and ε. Apply Cauchy's criterion to see uniform convergence on C.)

8) *(6 points)* Prove the following theorem of Landau: if a Dirichlet series $\mathcal{D}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has nonnegative real coefficients (i.e. $a_n \geqslant 0$ for all $n \in \mathbf{N}$), and the boundary line of its convergence domain is $\{s \in \mathbf{C} : \Re s = \sigma_0\}$, then it cannot be continued holomorphically to any neighborhood of σ_0. (*Hint:* show that by a renormalization of the coefficients, we can assume $\sigma_0 = 0$. Assume, by contradiction, that $\sum_{n=1}^{\infty} a_n n^{-s}$ extends to a small neighborhood of 0 as a holomorphic function $\alpha(s)$. Take a disc centered at 1 of radius slightly larger than 1 on which $\alpha(s)$ is holomorphic. Write $\alpha(s)$ in power series around 1 as $\alpha(s) = \sum_{k=0}^{\infty} c_k (s-1)^k$. Compute c_k in terms of a_n, then write $s = -\delta$ for some small $\delta > 0$ where the power series of $\alpha(s)$ is assumed to be convergent. Conclude contradiction by showing that this would imply that $\mathcal{D}(-\delta)$ is also convergent.)