## HOMEWORK SET 2

4) *(4 points)* Using that for all $0 < a < b < \infty$,
$$\sum_{x^a \leqslant p \leqslant x^b} \frac{1}{p} = \log \frac{b}{a} + o(1), \qquad x \to \infty,$$
prove that
$$\sum_p \frac{1}{p} f\left(\frac{\log p}{\log x}\right) = \int_0^\infty f(t)\frac{dt}{t} + o(1), \qquad x \to \infty$$
for any function $f : \mathbf{R}_+ \to \mathbf{C}$ which is compactly supported and Riemann integrable. (*Hint:* use upper and lower approximate sums which appear in the definition of Riemann integrability.)

5) *(4 points)* Let $f : \mathbf{N} \to \mathbf{C}$ be a function satisfying
$$f(m)f(n) = \sum_{d \mid \gcd(m,n)} f(mn/d^2), \qquad f(n) \ll n^{o(1)}, \qquad f(1) \neq 0.$$
Prove that the Dirichlet series
$$D(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$
is absolutely convergent for $s > 1$, and that it has the absolutely convergent Euler product
$$D(s) = \prod_p \left((1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})\right)^{-1}$$
for some $\alpha_p, \beta_p$, where $\alpha_p$, $\beta_p$ depend only on $f(p)$, and $\alpha_p \beta_p = 1$. (*Hint:* try to figure out from $f(p)$ what $\alpha_p, \beta_p$ can be.)

6) *(4 points)* Let $d(n)$ stand for the number of positive divisors of $n$. Prove that for any $x \geqslant 2$,
$$\sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + o(x).$$
(*Hint:* apply Dirichlet's hyperbola method as follows. First, write $\sum_{n \leqslant x} d(n)$ as $\sum_{d \leqslant x} \sum_{m \leqslant x/d} 1$. Separate $d \leqslant \sqrt{x}$ and $d > \sqrt{x}$. In the first type, compute directly, in the second type, change the order of the $d$-summation and the $m$-summation.)