HOMEWORK SET 1

1) *(2 points)* Prove by an example that convergence of infinite products does not imply their absolute convergence.
2) *(4 points)* Prove that, for any $x \geqslant 2$,
$$\prod_{p \leqslant x} p \leqslant 4^x.$$
(*Hint:* assume that x is an integer. Check the statement for $x \leqslant 10$. For $x > 10$ even, apply induction trivially. For $x > 10$ odd, apply induction together with the observation that $\binom{x}{(x-1)/2}$ is divisible by all primes $(x+1)/2 < p \leqslant x$.)
3) The goal of this problem is to prove the lower bound in Chebyshev's theorem. Consider, for any $n \in \mathbf{N}$, the canonical form of the binomial coefficient
$$\binom{2n}{n} = \prod_{p \leqslant \sqrt{2n}} p^{\alpha_p} \prod_{\sqrt{2n} < p \leqslant 2n/3} p^{\beta_p} \prod_{2n/3 < p \leqslant n} p^{\gamma_p} \prod_{n < p \leqslant 2n} p^{\delta_p}.$$
   a) *(1 point)* Prove that $\delta_p = 1$, $\gamma_p = 0$ for all p in the indicated ranges.
   b) *(1 point)* Prove that $0 \leqslant \beta_p \leqslant 1$ for all p in the indicated range. Conclude, using the previous problem that
$$\prod_{p \leqslant 2n/3} p \leqslant 4^{2n/3}.$$
   c) *(2 points)* Prove that $p^{\alpha_p} \leqslant 2n$ for all p in the indicated range.
   d) *(2 points)* Prove that $\prod_{n<p\leqslant 2n} \geqslant c^n$ for some $c > 1$. Conclude the lower bound $\sum_{n \leqslant x} \Lambda(n) \gg x$ in Chebyshev's theorem.