Combinatorial Number Theory
Homework 4

13. *(2 points)* Let G_1 and G_2 be finite abelian groups, $A_1 \subseteq G_1$, $A_2 \subseteq G_2$, and let $\phi : A_1 \to A_2$ be an F_8-isomorphism. Prove that ϕ induces an F_2-isomorphism between $2A_1 - 2A_1$ and $2A_2 - 2A_2$.

14. *(4 points)* Let $A \subseteq \mathbf{Z}$ be a subset of integers containing n elements. Prove that there is an F_2-isomorphic image of A in the interval $[0, 8^n]$. (*Hint:* As a preparation, assume that $B \subseteq \mathbf{F}_p$ satisfies $p > V^{|B|}$ for some positive integer V. Show that there is an $x \in \mathbf{F}_p^\times$ such that for all $b \in B$, there is an integer j_b with $|j_b| < p/V$ satisfying $xb = j_b$ in $\mathbf{F}_p$. Now return to the original problem, and take the minimal k such that A has an F_2-isomorphic image in $[0, k]$ (show that there is such). Now take a prime $2k < p < 4k$ and apply the preparatory statement with $V = 8$.)

15. *(3 points)* Prove that in the previous exercise, the bound 8^n cannot be improved below 2^{n-2}.

16. *(4 points)* Show by an example that the Szemerédi–Trotter theorem cannot hold over a finite field. (*Hint:* Start out from the projective plane over the finite field.)