Combinatorial Number Theory
Homework 3

9. *(2 points)* Show an example of a finite set $A \in \mathbf{Z}$ satisfying $|A-A| > |A+A|$. Show an example of a finite set $A \in \mathbf{Z}$ satisfying $|A+A| > |A-A|$. (*Hint:* There are examples contained in $\{0, \pm 3, \pm 4, \pm 5, \pm 7\}$.)

10. *(3 points)* Show that there is a constant $c > 1$ with the following properties. There are arbitrarily large finite sets $A \subset \mathbf{Z}$ satisfying $|A-A| > |A+A|^c$ and there are arbitrarily large finite sets $A \subset \mathbf{Z}$ satisfying $|A+A| > |A-A|^c$. (*Hint:* Use the examples from the previous exercise and apply the tensor power trick. Be careful: you have to end up with subsets of $\mathbf{Z}$.)

11. *(2 points)* Let A, B, C be finite subsets of an abelian group. Show that

$$E(A, B \cup C)^{1/2} \leqslant E(A,B)^{1/2} + E(A,C)^{1/2}.$$

(*Hint:* Use the triangle-inequality for the l^2-norm.)

12. *(7 points)* In an abelian group, we say that a set $\{a_1, \ldots, a_r\}$ is a Sidon set, if $a_1 + a_2 = a_3 + a_4$ is satisfied only when $\{a_1, a_2\} = \{a_3, a_4\}$. Show that the largest Sidon set in $\{1, \ldots, n\}$ has size $n^{1/2} + O(n^{1/3})$. (*Hint:* For a Sidon set $1 \leqslant a_1 < a_2 < \ldots < a_r \leqslant n$, estimate from both ways the sum of all differences $a_j - a_i$ with $i + 1 \leqslant j \leqslant i + u$, where u is a parameter depending on n. Compare the upper and lower bounds to deduce an upper bound for r, then optimize u. To find a large Sidon set in $\{1, \ldots, n\}$, show that for a prime p and a primitive root g modulo p, the set $\{(t, g^t) \in \mathbf{Z}_{p-1} \times \mathbf{Z}_p : 1 \leqslant t \leqslant p-1\}$ is a Sidon set. Use the Chinese remainder theorem and a Hoheisel type bound about primes in short intervals.)