Combinatorial Number Theory
Homework 2

6. *(4 points)* For any $k \geqslant 1$, construct an asymptotic basis B of order k satisfying $|B \cap [1, n]| = \Theta(n^{1/k})$ for $n \geqslant 1$. (*Hint:* Use the base k representation of the integers.)

7. *(4 points)* Let $x_1, \ldots, x_k$ be positive integers not exceeding n such that all the 2^k sums $\sum_{i \in S} x_i$, $S \subseteq \{1, \ldots, k\}$ are distinct. Prove that

$$k \leqslant \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1).$$

(*Hint:* Pick S at random so that $X = \sum_{i \in S} x_i$ is a random variable. Give an upper and a lower bound on the probability that X is close to its expectation, both in terms of k. For the lower bound, use that all the subsums are distinct, i.e. X takes each of its values by probability 2^{-k}. For the upper bound, apply Chebyshev's inequality. Compare the bounds and deduce that k cannot be large.)

8. *(4 points)* Let $c > 0$ be arbitrary. Take a random graph on n vertices such that each of the $\binom{n}{2}$ potential edges are present with probability c/n (independently). Prove that the resulting graph is triangle-free with probability tending to $\exp(-c^3/6)$ (as n goes to infinity). (*Hint:* For the lower bound, use the FKG inequality specialized to indicator functions of events: if $A_1, \ldots, A_n$ are events, all increasingly depending on some boolean variables, then $\mathbf{P}(A_1 \cap \ldots \cap A_n) \geqslant \mathbf{P}(A_1) \cdot \ldots \cdot \mathbf{P}(A_n)$. For the upper bound, use Janson's inequality in a modified version: in the proof of the lecture, we took $t = T/\Delta$ to obtain a general bound, but for $T = \mathbf{E}(X)$, this choice is not optimal.)