Combinatorial Number Theory
Homework 1

1. *(4 points)* Let $A_1, \ldots, A_n$ be nonempty subsets of the finite abelian group G. Prove that there are elements $g_1, \ldots, g_n \in G$ satisfying

$$1 - \frac{|(g_1 + A_1) \cup \ldots \cup (g_n + A_n)|}{|G|} \leqslant \left(1 - \frac{|A_1|}{|G_1|}\right) \cdot \ldots \cdot \left(1 - \frac{|A_n|}{|G_n|}\right).$$

   (*Hint:* Pick $g_1, \ldots, g_n \in G$ at random.)

2. *(2 points)* Let A be a nonempty subset of the finite abelian group G. Prove that G can be covered with

$$\left\lceil \frac{|G| \log |G|}{|A|} \right\rceil$$

   translates of A. (*Hint:* Use the result of the previous problem.)

3. *(6 points)* Let n be a sufficiently large integer. Show that one can color the elements of the set $\{1, \ldots, n\}$ with red and blue such that the following three properties simultaneously hold:

   (a) there is a monochromatic arithmetic progression of length $\log n/10$;

   (b) there is no monochromatic arithmetic progression of length $10 \log n$;

   (c) in every arithmetic progression, the number of red and blue elements differ by at most $10\sqrt{n \log n}$.

   (*Hint:* Show that in a random coloring, each of these properties are very likely. For part (a), focus on a set of many disjoint arithmetic progression. For part (c), the approximation $\binom{2m}{m-k}/\binom{2m}{m} \approx e^{-k^2/m}$ might be useful (if you use it, a rigorous form is needed).)

4. *(5 points)* For $B \subseteq \mathbf{N}$, denote by $r_{k,B}(n)$ the number of ways n can be written as the sum of k elements of B. Show that for a positive integer m, it is impossible that $r_{k,B}(n) = m$ holds for all but finitely many n. (*Hint:* Prove by contradiction, and for such a set B, consider the generating function $F(z) = \sum_{b \in B} z^b$ on $|z| < 1$. From the conditions, prove conflicting bounds on $\int_0^1 |F(re^{2\pi it})|^2 dt$ if $r < 1$ is close enough to 1. For the upper bound, prove $F(z) = O(|1 - z|^{-1/k})$, and for the lower bound, prove $B \cap [1, n] = \Omega(n^{1/k})$.)

5. *(4 points)* Assume $t_1, \ldots, t_n$ are jointly independent random boolean variables such that for all $1 \leqslant j \leqslant n$, $\mathbf{P}(t_j = 0), \mathbf{P}(t_j = 1) > 0$. Assume $X = X(t_1, \ldots, t_n)$ and $Y = Y(t_1, \ldots, t_n)$ are both increasing functions of these variables. Prove that $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y)$ if and only if X and Y depend on disjoint subsets of $\{t_1, \ldots, t_n\}$. (*Hint:* Follow carefully our inductive proof of the FKG inequality and show that X and Y cannot simultaneously depend on t_n.)