Real functions and measures

Midterm exam

Earlier semesters (2022 Fall + 2023 Spring)

Regulations

1. You have 120 minutes for the exam.
2. The exam is closed books, no calculator, no internet. Only pen/pencil, rubber and scratch paper can be used.
3. Partial scores can be earned on any problem.

Problem 1. Let $U \subseteq \mathbf{R}^d$ be a nonempty open set. Prove that $\lambda_d(U) > 0$.

Problem 2. Let $C := [0,1]^3$ be the closed unit cube in the 3-dimensional space. Prove that $\lambda_3(C) = 1$.

Problem 3. Let X be a countably infinite set. What is the σ-algebra generated by the collection of all finite subsets of X?

Problem 4. Assume $(X, \mathcal{A})$ is a measurable space. Let $f_1, f_2, \ldots : X \to (0,1)$ be a sequence of measurable functions (i.e. f_n is measurable for every $n \in \mathbf{N}$, and $0 < f_n(x) < 1$ for every $n \in \mathbf{N}$ and every $x \in X$). Prove that

$$\left\{ x \in X : \sum_{n=1}^{\infty} f_n(x) < 2022 \right\}$$

is a measurable set.

Problem 5. Assume that $A \subseteq (0,1)$ satisfies that $\lambda(A) > 0$. Prove that there exists some $a \in (0,1)$ such that

$$\lambda(A \cap (0,a)) > 0 \qquad \text{and} \qquad \lambda(A \cap (a,1)) > 0$$

simultaneously hold.

Problem 1. Let $C \subseteq \mathbf{R}^d$ be a compact set. Prove that $\lambda_d(C) < \infty$.

Problem 2. Let $(\mathbf{R}, \mathcal{L}, \lambda)$ be the one-dimensional Lebesgue measure space (with $\mathcal{L}$ standing for the σ-algebra of Lebesgue measurable sets and λ for the Lebesgue measure). Define the function $f : \mathbf{R} \to \mathbf{R}$ as $f(x) := x$ for any $x \in \mathbf{R}$. Prove that f is measurable, and calculate the integral $\int_{\mathbf{R}} f \, d\lambda$ or prove that the integral $\int_{\mathbf{R}} f \, d\lambda$ is not defined.

Problem 3. Is it true that

$$\lambda_d(A \cup B) = \lambda_d(A) + \lambda_d(B)$$

holds for all pairs of disjoint sets $A, B \subseteq \mathbf{R}^d$?

Problem 4. Let $(\mathbf{R}, \mathcal{L}, \lambda)$ be the one-dimensional Lebesgue measure space (with $\mathcal{L}$ standing for the σ-algebra of Lebesgue measurable sets and λ for the Lebesgue measure). Define the function sequence $f_1, f_2, \ldots$ as

$$f_n(x) := \begin{cases} 1 - x^n, & x \in [0,1] \\ 0, & x \notin [0,1]. \end{cases}$$

Prove that

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n \, d\lambda$$

exists and compute its value. (You can take for granted that every f_n is a measurable function, you do not have to prove it.)

Problem 5. Let $(X, \mathcal{A})$ be a measruable space, and assume that $f_1, f_2, \ldots : X \to \overline{\mathbf{R}}$ is a sequence of measurable functions. Prove that

$$\left\{x \in X : \lim_{n\to\infty} f_n(x) = 0\right\} \in \mathcal{A}.$$