Real functions and measures

Final exam

2022 Fall

Regulations

1. You have 120 minutes for the exam.
2. The exam is closed books, no calculator, no internet. Only pen/pencil, rubber and scratch paper can be used.
3. Partial scores can be earned on any problem.

Problem 1. In this problem, our measure space is $(\mathbf{R}, \mathcal{L}, \lambda)$. Consider the function defined as $f(x) := |x|$. Compute $\int_{\mathbf{R}} f \, d\lambda$. (You can use without proof that f is measurable.)

Problem 2. In this problem, our measure space is $(\mathbf{R}^2, \mathcal{L}_2, \lambda_2)$. Consider the square S defined as

$$S := \{(x, y) \in \mathbf{R}^2 : |x| + |y| \leqslant 1\}.$$

(This is the closed square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.) Compute $\lambda_2(S)$.

Problem 3. Let $(X, \mathcal{A})$ be a measurable space, and assume that $f : X \to \mathbf{R}$ is a measurable function such that $0 < f(x) < \infty$ for all $x \in X$. Define then the function $g : X \to \mathbf{R}$ via the formula

$$g(x) := \frac{1}{f(x)}.$$

Prove that g is also a measurable function.

Problem 4. In this problem, our measure space is $(\mathbf{R}, \mathcal{L}, \lambda)$. Consider the function sequence

$$f_n := \mathbf{1}_{[n, n+\frac{1}{n}]}.$$

Compute $\lim_{n\to\infty} \int_{\mathbf{R}} f_n \, d\lambda$ and $\int_{\mathbf{R}} \lim_{n\to\infty} f_n \, d\lambda$, and decide if there exists an integrable dominant for the collection $(f_n)_{n\in\mathbf{N}}$.

Problem 5. Assume μ and ν are two measures on the measurable space $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ which satisfy that $\mu(U) = \nu(U)$ holds for any open set U. Does it follow that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}(\mathbf{R})$?