Midterm Exam Regulations – read them carefully

1. You have a continuous 120 minutes to take the exam.
2. The exam is closed books, no notes, no calculators, no internet. You can use only clean sheets, pens, pencils.
3. Keep in mind that partial scores can be earned in any problem. So if you cannot solve a problem, but you have some thoughts that you consider a good approach, hand them in. This particularly applies to the last problem, which is significantly harder than the preceding ones.

1. Define on $\mathbf{N}$ the collection τ as follows: for any $A \subseteq \mathbf{N}$,

$$A \in \tau \qquad \text{if and only if} \qquad A = \emptyset \text{ or } A^c \text{ is finite.}$$

   Prove that $(\mathbf{N}, \tau)$ is a topology.

2. Let $(X, \mathcal{A})$ be a measurable space. Assume that μ, ν are two measures on $(X, \mathcal{A})$ such that for any $A \in \mathcal{A}$,

$$\mu(A) \geqslant \nu(A).$$

Prove that for any measurable function $f \geqslant 0$,

$$\int_X f \, d\mu \geqslant \int_X f \, d\nu.$$

3. Consider the measure space $(\mathbf{N}, 2^{\mathbf{N}}, \#)$, where $\#$ denotes the counting measure. Consider the function f defined as

$$f(n) = \frac{(-1)^n}{2^n}, \qquad n \in \mathbf{N}.$$

Calculate $\int_{\mathbf{N}} f \; d\#$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, \ldots : X \to \mathbf{R}$ be a sequence of negative functions, i.e. $f_n(x) < 0$ for any $n \in \mathbf{N}$ and $x \in X$. Prove that

$$\limsup_{n\to\infty} \int_X f_n \, d\mu \leqslant \int_X \limsup_{n\to\infty} f_n \, d\mu.$$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to \mathbf{R}$ be a measurable function such that $f \geqslant 0$ and $\int_X f \, d\mu < \infty$. Prove that there exists some $\delta > 0$ with the following property: if $A \subseteq X$ is measurable and $\mu(A) < \delta$, then $\int_A f \, d\mu < 1$.