Midterm Exam Regulations – read them carefully

1. You have a continuous 120 minutes to take the exam.
2. The exam is closed books, no notes, no calculators, no internet. You can use only clean sheets, pens, pencils.
3. Keep in mind that partial scores can be earned in any problem. So if you cannot solve a problem, but you have some thoughts that you consider a good approach, hand them in. This particularly applies to the last problem, which is significantly harder than the preceding ones.

1. What is the smallest σ-algebra $\mathcal{A}$ on $\mathbf{R}$ which satisfies that

$$f(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0 \end{cases}$$

is measurable with respect to $\mathcal{A}$?

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $A_1, A_2, \ldots \in \mathcal{A}$. Prove that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu(A_n).$$

3. Let $(X, \mathcal{A})$ be a measurable space, and $f_1, f_2, \ldots : X \to \mathbf{C}$ be measurable functions. Prove that

$$\{x \in X : f_1(x), f_2(x), \ldots \text{ is a bounded sequence}\}$$

is measurable with respect to $\mathcal{A}$.

4. Consider the measure space $(\mathbf{N}, 2^{\mathbf{N}}, \mu)$, where μ is the counting measure. Let

$$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } n < x \leqslant 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\lim_{n\to\infty}(\int_{\mathbf{N}} f_n \; d\mu)$, $\int_{\mathbf{N}}(\lim_{n\to\infty} f_n) \; d\mu$, and decide if there exists an integrable majorant for the collection $f_1, f_2, \ldots$.

5. For any differentiable function $f : \mathbf{R} \to \mathbf{R}$, we set $Lf = f'(0)$, i.e. L evaluates the derivative of f at 0. Since differentiation is a linear operator, L is a linear map from the vector space of differentiable functions to $\mathbf{R}$ (you can use this fact without proving it). Decide if there exist Borel measures μ, ν on $\mathbf{R}$ such that

$$Lf = \int_{\mathbf{R}} f \; d\mu - \int_{\mathbf{R}} f \; d\nu$$

holds for any differentiable function $f : \mathbf{R} \to \mathbf{R}$.