Final Exam Regulations – read them carefully

1. You have a continuous 120 minutes (no breaks) to take the exam.

2. The exam is closed books, no notes, no calculators, no internet. You can use only clean sheets, pens, pencils.

3. Keep in mind that partial scores can be earned in any problem. So if you cannot solve a problem, but you have some thoughts that you consider a good approach, hand them in.

1. Prove that if U is a nonempty, open subset of $\mathbf{R}^d$. then $\lambda_d(U) > 0$.

   Solution. Since U is nonempty, we can take some $x = (x_1, \ldots, x_d) \in U$. Since U is open, for some $r > 0$, $B(x, r) \subseteq U$. Then

$$V = \left(x_1 - \frac{r}{d}, x_1 + \frac{r}{d}\right) \times \ldots \times \left(x_d - \frac{r}{d}, x_d + \frac{r}{d}\right) \subseteq B(x, r),$$

   since any point of V differs from x by less than r/d in each coordinate, hence by the triangle-inequality, altogether by less than r. Then $V \subseteq U$, and using that V is the product of intervals of positive length, its Lebesgue measure is the product of its edge lengths (proven in class), that is,

$$\lambda_d(U) \geqslant \lambda_d(V) = \left(\frac{2r}{d}\right)^d > 0,$$

   and the proof is complete.

2. Let λ_2 be the 2-dimensional Lebesgue measure on the plane $\mathbf{R}^2$. Consider the triangle

$$T = \left\{(x, y) \in \mathbf{R}^2 : x \geqslant 0, y \geqslant 0, x + y \leqslant 1\right\}.$$

Prove that $\lambda_2(T) = 1/2$.

Preliminary. In the solutions, we will use the term Lebesgue measure for λ_2, and will frequently use that the Lebesgue measure of a rectangle is the product of its edges (proven in class).

Solution 1. Let us denote by T' the closed triangle which completes T to the unit square $[0, 1]^2$, i.e.

$$T' = \{(x, y) : x \leqslant 1, y \leqslant 1, x + y \geqslant 1\}.$$

Let $D = T \cap T'$ be the diagonal of the square. Then $\lambda_2(D) = 0$ (proper subspaces and their shifted copies have measure 0: mentioned in class, details were left to an elementary exercise in the notes). Also observe that $T' \setminus D$ is a congruent copy of $T \setminus D$, namely,

$$T' \setminus D = (T \setminus D) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + (1, 1),$$

hence $\lambda_2(T \setminus D) = \lambda_2(T' \setminus D)$ (we proved in the class how linear transformations and shifts change the Lebesgue measure). Then in the pairwise disjoint union $[0, 1]^2 = (T \setminus D) \cup (T' \setminus D) \cup D$, the left-hand side admits Lebesgue measure 1, so does the right-hand side, where the last term gives 0, and the first two terms contribute the same value, hence both are 1/2. Then

$$\lambda_2(T) = \lambda_2(T \setminus D) + \lambda_2(D) = \frac{1}{2} + 0 = \frac{1}{2},$$

and the proof is complete.

Solution 2. *(In case one does not remember our results about the relation of linear transformations, shifts and the Lebesgue measure. This is a little more computational. There will be some rectangles defined, draw them for, say, $n = 4$, because the formulae are somewhat complicated, but the idea of them is quite visible.)* Observe that for any $n \in \mathbf{N}$, T is covered with the following union of rectangles:

$$\bigcup_{j=0}^{n-1} \left[\frac{j}{n}, \frac{j+1}{n}\right] \times \left[0, 1 - \frac{j}{n}\right].$$

The Lebesgue measure of the jth such rectangle is $(n - j)/n^2$, which shows that

$$\lambda_2(T) \leqslant \sum_{j=0}^{n-1} \frac{n - j}{n^2} = \frac{n^2 + n}{2n^2} = \frac{1}{2} + \frac{1}{2n}.$$

Also, T contains the pairwise disjoint union of rectangles

$$\bigcup_{j=0}^{n-1} \left(\frac{j}{n}, \frac{j+1}{n}\right) \times \left(0, 1 - \frac{j+1}{n}\right).$$

The Lebesgue measure of the jth such rectangle is $(n - j - 1)/n^2$, which shows that

$$\lambda_2(T) \geqslant \sum_{j=0}^{n-1} \frac{n - j - 1}{n^2} = \frac{n^2 - n}{2n^2} = \frac{1}{2} - \frac{1}{2n}.$$

Since the two estimates hold for any $n \in \mathbf{N}$, we can take $n \to \infty$ in both, resulting that $\lambda_2(T) = 1/2$.

Remark. One can proceed by computing the limit of $L_n(\mathbf{1}_T)$, where L_n's are the positive, linear functionals used in the class to define the Lebesgue measure. It leads to a similar, but slightly more complicated calculation than the one in Solution 2.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Assume that the measurable function $f : X \to \mathbf{C}$ satisfies $f \in L^1(X)$ and $|f(x)| < 1$ for almost every $x \in X$. (Recall, this means that $\mu(\{x \in X : |f(x)| \geqslant 1\}) = 0$.) Prove that

$$\lim_{n\to\infty} \int_X f^n \, d\mu = 0.$$

(To clarify, for any $n \in \mathbf{N}$, the function f^n is defined as $f^n(x) := (f(x))^n$.)

Solution. Let us denote by S the set where $|f| < 1$, by assumption, $\mu(X \setminus S) = 0$, hence for any $n \in \mathbf{N}$,

$$\int_X f^n \, d\mu = \int_S f^n \, d\mu.$$

Note also that on S, $|f|$ is an integrable majorant for the sequence $f, f^2, f^3, \ldots$, hence dominated convergence applies. Also, on S, f^n tends pointwise to 0. Summing up,

$$\lim_{n\to\infty} \int_X f^n \, d\mu = \lim_{n\to\infty} \int_S f^n \, d\mu = \int_S \lim_{n\to\infty} f^n \, d\mu = \int_S 0 \, d\mu = 0,$$

and the proof is complete.

4. Define the positive, linear functional $L : C_c(\mathbf{R}) \to \mathbf{C}$ as

$$Lf := \int_{-1}^{1} f(x)\ dx + f(2021), \qquad f \in C_c(\mathbf{R}).$$

Let μ be the measure associated to L via the Riesz representation theorem. Prove that $\mu(\mathbf{R}) = 3$. (In $\mathbf{R}$, we consider the standard, euclidean topology. The integral in the definition of L is the Riemann integral, you can use without proof that it makes sense for every compactly supported, continuous function. You can use without proof that L is indeed positive and linear.)

Preliminary. For any $a, b \in \mathbf{R}$ with $a \leqslant b$, define the function

$$H_{a,b} = (\mathrm{id} - (a-1)) \cdot \mathbf{1}_{(a-1,a)} + \mathbf{1}_{[a,b]} + (b+1-\mathrm{id})\mathbf{1}_{(b,b+1)}.$$

Again, draw an image: the formula is complicated, but the content is simple, the function is constant 0 up to $a-1$ and above $b+1$, constant 1 between a and b, and in the remaining segments $(a-1, a)$ and $(b, b+1)$, it connects the constant pieces linearly. Note that $H_{a,b} \in C_c(\mathbf{R})$ for any choice of a, b.

Solution 1. The set $\mathbf{R}$ is itself open, so from the definition of μ on open sets in the proof of the Riesz representation theorem, we know that

$$\mu(\mathbf{R}) = \sup\{Lf : f \in C_c(\mathbf{R}), 0 \leqslant f \leqslant 1\}.$$

For any function f in the supremum, we have

$$Lf = \int_{-1}^{1} f(x)\ dx + f(2021) \leqslant \int_{-1}^{1} 1\ dx + 1 = 3.$$

Also, for $H_{-1,2021} \in C_c(\mathbf{R})$,

$$LH_{-1,2021} = \int_{-1}^{1} H_{-1,2021}(x)\ dx + H_{-1,2021}(2021) = \int_{-1}^{1} 1\ dx + 1 = 3,$$

and the proof is complete.

Solution 2. *(In case one does not remember how we defined μ in the proof of the Riesz representation theorem.)* Let $n \geqslant 2022$. Then

$$\begin{aligned}\mu([-n,n]) &= \int_{\mathbf{R}} \mathbf{1}_{[-n,n]}\ d\mu \leqslant \int_{\mathbf{R}} H_{-n,n}\ d\mu = LH_{-n,n} \\ &= \int_{-1}^{1} H_{-n,n}(x)\ dx + H_{-n,n}(2021) = \int_{-1}^{1} 1\ dx + 1 = 3.\end{aligned}$$

Also,

$$\begin{aligned}\mu([-n,n]) &= \int_{\mathbf{R}} \mathbf{1}_{[-n,n]}\ d\mu \geqslant \int_{\mathbf{R}} H_{1-n,n-1}\ d\mu = LH_{1-n,n-1} \\ &= \int_{-1}^{1} H_{1-n,n-1}(x)\ dx + H_{1-n,n-1}(2021) = \int_{-1}^{1} 1\ dx + 1 = 3.\end{aligned}$$

Then $\mu([-n,n]) = 3$ for $n \geqslant 2022$. We have proven in class that for monotone increasing union of sets, the measure of the union equals the limit of the individual measures, i.e.

$$\mu(\mathbf{R}) = \mu\left(\bigcup_{n=2022}^{\infty} [-n,n]\right) = \lim_{n\to\infty} \mu([-n,n]) = 3,$$

and the proof is complete.

5. Does there exist a sequence of Lebesgue measurable functions $f_1, f_2, \ldots$ on the real line which satisfies simultaneously

- that $f_n \in L^p(X)$ for any $n \in \mathbf{N}$ and any $1 \leqslant p \leqslant \infty$;
- that for every $1 \leqslant p < \infty$,

$$\lim_{n\to\infty} \|f_n\|_{L^p} \to 0;$$

- and that

$$\lim_{n\to\infty} \|f_n\|_{L^\infty} \to \infty?$$

Solution. Yes, we construct such a sequence. Let

$$f_n = n \cdot \mathbf{1}_{[0,1/n^n]},$$

and we claim they do the job.

For any $1 \leqslant p < \infty$,

$$\|f_n\|_{L^p} = \left(\int_{[0,1/n^n]} n^p \; d\lambda \right)^{1/p} = (n^{p-n})^{1/p} = n^{1-n/p}.$$

In particular, each f_n is in L^p for any $1 \leqslant p < \infty$, and the limit, as n tends to ∞, is 0.

Also,

$$\|f_n\|_{L^\infty} = n.$$

In particular, each f_n is in L^∞, and the limit, as n tends to ∞, is ∞.