Final Exam Regulations – read them carefully

1. On the next page, the actual Final Exam starts. Read all the instructions now, but please, **DO NOT** scroll down to the next page until you are ready to take the exam.
2. You have a continuous 120 minutes (no breaks) to take the exam. Please, do not make any corrections after this 120-minute window.
3. The exam is closed books, no notes, no calculators, no internet. You can use only clean sheets, pens, pencils.
4. You are not allowed to discuss the exam with anyone before you receive an e-mail from me allowing that.
5. Once done, take a scan or a photo of your papers you want to hand in, and send them to me in e-mail. (If possible, send me everything in a single pdf file, where the filename contains your name, e.g. if I took the exam, I would send `magapeter_rfm_final.pdf`. Of course, images in jpg or any other standard file format are accepted.)
6. Please, also sign the statement of honor on this frontpage, and send it to me together with your exam (preferably in the same file).
7. First submission deadline is May 11 (Tuesday), submit before midnight in your time zone. If you cannot make it by this deadline, send me an e-mail by May 10 (Monday), and we negotiate the "individual" deadline. Please, go for this second option only for serious reasons.
8. Keep in mind that partial scores can be earned in any problem. So if you cannot solve a problem, but you have some thoughts that you consider a good approach, hand them in.

Statement of Honor

I pledge my honor that I do not give or receive any inappropriate aid for this assignment.

signature

1. Let C be a compact subset of $[0, 1]$. Prove that if $C \neq [0, 1]$, then $\lambda_1(C) < 1$.

2. Let f_n be a sequence of continuous functions from $[-1,1]$ to $\mathbf{R}$ satisfying that $-1 \leqslant f_n(x) \leqslant 1$ (for any $n \in \mathbf{N}$ and $x \in [-1,1]$) and $f_n(x) \to 0$ (for any $x \in [-1,1]$ as $n \to \infty$). Does this imply that

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\, dx = 0\ ?$$

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that the following two statements are equivalent:

   - $\mu(X) < \infty$;
   - for every bounded measurable function $f : X \to \mathbf{C}$, we have $f \in L^1(X)$.

   (To clarify: you have to prove two things. One is that if $\mu(X) < \infty$, then $f \in L^1(X)$ for every bounded measurable f. The other one is that if $f \in L^1(X)$ for every bounded measurable f, then $\mu(X) < \infty$. Recall that $f \in L^1(X)$ means that $\int_X |f| \, d\mu < \infty$.)

4. Define the function f as $f(x) := x$ for every $x \in \mathbf{R}$ (i.e. f is the identity function). What is

$$\int_{\mathbf{R}} f \, d\lambda_1 \ ?$$

(You can use without proof that f is measurable with respect to the Lebesgue measure. This is a simple consequence of that f is continuous.)

5. Let μ be a regular Borel measure on a compact Hausdorff space X with $\mu(X) = 1$. Prove that there exists a compact set $C \subseteq X$ such that

   - $\mu(C) = 1$;
   - if U is an open set satisfying $U \cap C \neq \emptyset$, then $\mu(U) > 0$.

   (Recall that a Borel measure μ is regular, if for any Borel set A,

   $$\mu(A) = \sup\{\mu(C) : A \supseteq C \text{ compact}\} \qquad \text{and} \qquad \mu(A) = \inf\{\mu(U) : A \subseteq U \text{ open}\}$$

   both hold.)