1. (a) State the Chinese remainder theorem. **(2 points)**
   (b) Compute $13^{24} \bmod 56$. **(4 points)**

2. (a) Define Hurwitz quaternions. **(2 points)**

   (b) Assume $\alpha = a + bi + cj + dk$ is a Hurwitz quaternion such that $N(\alpha)$ is an even integer. Prove that $a, b, c, d \in \mathbf{Z}$. **(4 points)**

3. (a) State Gauss's lemma about primitive polynomials. **(2 points)**

   (b) Let $p(x), q(x) \in \mathbf{Z}[x]$. Prove that if one of them is not primitive, then their product is also not primitive. **(4 points)**

4. (a) What is Pell's equation? **(2 points)**

   (b) Let $d > 0$ be a square-free integer. Assume $0 < x_1 < \ldots < x_n$, $0 < y_1 < \ldots < y_n$ satisfy $x_j^2 - dy_j^2 = 1$ for all $1 \leq j \leq n$. Prove that $x_n > \sqrt{d}^n$. **(4 points)**