Mock Midterm Exam
Solutions

In the exam, there are four problems, each is worth six points. Note that complete arguments are required. Please make sure that you write your name on every page.

1. Compute $\tau_1(2016)$. **(6 points)**

   Solution. The number 2016 has canonical form

   $$2016 = 2^5 \times 3^2 \times 7.$$

   As we learned it in the lecture, the function τ_1 is multiplicative, therefore

   $$\tau_1(2016) = \tau_1(2^5) \times \tau_1(3^2) \times \tau_1(7).$$

   If p is a prime number, then the divisors of the numbers p^k are $1, p, \ldots, p^k$, and their sum is

   $$1 + p + \ldots + p^k = \frac{p^{k+1} - 1}{p - 1}.$$

   Applying this to 2^5, 3^2 and 7, we obtain

   $$\tau_1(2016) = \tau_1(2^5) \times \tau_1(3^2) \times \tau_1(7) = 63 \times 13 \times 8 = 6552.$$

2. Prove that if $n > 42$ is an integer, then there exist positive integers x, y such that the equation

$$6x + 7y = n$$

holds. **(6 points)**

Solution. For any integer $n > 42$, the given equation is a linear diophantine equation in two variables x, y. We know from the lecture that $ax + by = c$ has solutions over the integers if $\gcd(a, b) \mid c$. Since $\gcd(6, 7) = 1$, and $1 \mid n$, we see there exist integer solutions.

Let x_0, y_0 be such a solution. We also know that the set of all solutions is parametrized then as

$$\{(x, y) = (x_0 + 7t, y_0 - 6t) : t \in \mathbf{Z}\}.$$

Then for any $t \in \mathbf{Z}$,

$$6x = 6x_0 + 42t, \qquad 7y = 7y_0 - 42t.$$

From this, we see that $6x$ and $7y$ are determined only modulo 42, therefore we may choose $6x$ such that it is in the interval $[1, 42]$. Then on the one hand, $6x > 0$, so $x > 0$. On the other hand, $7y = n - 6x \geq n - 42 > 0$, so $y > 0$.

3. Prove that if $p > 2$ is a prime, and a, b are primitive roots modulo p, then ab is not a primitive root modulo p. **(6 points)**

Solution. It suffices to show that ab is a quadratic residue modulo p, since then all its powers are quadratic resiudes as well, therefore ab does not generate the multiplicative group modulo p (quadratic non-residues are not generated).

Since a, b are primitive roots, they must be quadratic non-residues (by the same argument: the powers of a quadratic residue are all quadratic residues). Recall from the lecture that for any x coprime to p, we have

$$\left(\frac{x}{p}\right) \equiv x^{\frac{p-1}{2}} \bmod p.$$

Applying this,

$$\left(\frac{ab}{p}\right) \equiv (ab)^{\frac{p-1}{2}} = a^{\frac{p-1}{2}} b^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = (-1) \times (-1) = 1 \bmod p.$$

This shows ab is a quadratic residue modulo p.

4. Prove that there exist positive integers n satisfying $\tau_1(n) > 100n$. **(6 points)**

Solution. Choose M such that

$$1 + \frac{1}{2} + \ldots + \frac{1}{M} > 100.$$

This can be done, since the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, or we can refer to the lecture, where it was proved that the reciprocal sum of the primes is infinity (then obviously so is the reciprocal sum of all natural numbers).

Now take any n divisible by $M! = 1 \times 2 \times \ldots \times M$. Then n is divisible by all the numbers $1, 2, \ldots, M$ and so is by their divisor pairs $n, n/2, \ldots, n/M$ (and these are distinct positive numbers). Then

$$\tau_1(n) = \sum_{d|n} d \geq n + \frac{n}{2} + \ldots + \frac{n}{M} = n\left(1 + \frac{1}{2} + \ldots + \frac{1}{M}\right) > 100n,$$

and the proof is complete.