Mock Final Exam

1. (a) What is the conjugate of a gaussian integer? **(2 points)**

   (b) Prove that for any gaussian integers α, β, we have $\overline{\alpha} + \overline{\beta} = \overline{\alpha + \beta}$ and $\overline{\alpha} \times \overline{\beta} = \overline{\alpha \times \beta}$. **(4 points)**

2. (a) What is the Pell equation? State the structure theorem about its solutions. **(2 points)**
   (b) Give three solutions of the Pell equation $x^2 - 3y^2 = 1$ satisfying also $x, y > 0$. **(4 points)**

3. (a) In $\mathbf{Z}$, what is the definition of prime numbers (the definition we used in the class)? In $\mathbf{Z}$, what is the definition of irreducible numbers (the definition we used in the class)? What was proved about primes and irreducibles in $\mathbf{Z}$? **(2 points)**

   (b) Give all positive integes n such that $n^3 - 27$ is a prime number. (Take care: although n is positive, $n^3 - 27$ can be negative, and there are negative primes!) **(4 points)**

4. (a) State the Chinese remainder theorem. **(2 points)**

   (b) Prove that there exist a positive integer n such that none of $n+1, \ldots, n+100$ is square-free. **(4 points)**