Midterm Exam

In the exam, there are four problems, each is worth six points. Note that complete arguments are required. Please make sure that you write your name on every page.

1. Compute $\varphi(2016)$. **(6 points)**

   Solution. The number 2016 has canonical form

   $$2016 = 2^5 \times 3^2 \times 7.$$

   As we learned it in the lecture,

   $$\varphi(n) = n \prod_{j=1}^{r} \left(1 - \frac{1}{p_j}\right),$$

   if the prime divisors of n are $p_1, \ldots, p_r$. Then

   $$\varphi(2016) = 2016 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{6}{7} = 576.$$

2. Prove that for any positive integer n, there exists a positive integer $x \in [n, n+99]$ satisfying $x \equiv 2 \bmod 4$ and $x \equiv 3 \bmod 25$. **(6 points)**

Solution. By the Chinese remainder theorem, there is a residue class c modulo 100 such that $c \equiv 2 \bmod 4$ and $c \equiv 3 \bmod 25$. Since the numbers $n, n+1, \ldots, n+99$ are all distinct modulo 100, and their number is exactly 100, which is exactly the number of residue classes modulo 100, one of them is c modulo 100. Choosing that one for x, the proof is complete.

3. Prove that if $n > 2$, then $\varphi(n)$ is even. **(6 points)**

Solution. Assume that the canonical form of n is $n = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$. Since $n > 2$, at least one of the following two possibilities hold: n has an odd prime divisor; the exponent of 2 in the canonical form of n is at least 2. Indeed, if n has no odd prime divisor, then $n = 2^\alpha$, then $\alpha = 1$ would imply $n = 2$ which is excluded.

Then

$$\varphi(n) = n \prod_{j=1}^{r} \left(1 - \frac{1}{p_j}\right) = \prod_{j=1}^{r} \left(p_j^{\alpha_j} - p_j^{\alpha_j - 1}\right).$$

Now observe that if n has an odd prime divisor, say, p_j, then $p_j^{\alpha_j} - p_j^{\alpha_j - 1}$ is an even factor in the product above (since it is the difference of two odd numbers), implying that $\varphi(n)$ is even as desired.

Also, if $n = 2^\alpha$ with $\alpha \geq 2$, then $\varphi(n) = 2^{\alpha - 1}$, which is even. The proof is complete.

Alternative solution. Consider the coupling $a \leftrightarrow n - a$ on the residue classes coprime to n. Then each coprime residue class has a match, and we claim it is different from itself. Indeed,

$$a \equiv n - a \bmod n$$

holds if and only if $a \equiv 0$ or $n/2$ modulo n (the latter one only if n is even). Here, 0 is not a coprime residue class to n, and $n/2$ is also not a coprime residue class to n, since $n > 2$.

Now $\varphi(n)$ is twice the number of couples, which is then an even number.

4. Characterize by their canonical form the positive integers $n > 1$ satisfying $\tau_1(n) < n + \sqrt{n}$. (Note: you have to state and prove an 'if and only if' statement.) **(6 points)**

Solution. We claim that for integers $n > 1$, $\tau_1(n) < n + \sqrt{n}$ if and only if n is a prime, so its canonical form is n.

First, if n is a prime, then its only divisors are 1 and n. Then

$$\tau_1(n) = n + 1 < n + \sqrt{n}.$$

While if n is not a prime, then $n = ab$ for some $1 < a \leq b < n$. We claim that $b \geq \sqrt{n}$. Indeed, if $a \leq b < \sqrt{n}$, then $ab < \sqrt{n}^2 = n$, which is a contradiction. Now

$$\tau_1(n) \geq n + b + 1 > n + \sqrt{n},$$

which completes the proof.