Problems - 2012.11.14

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1. In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i-th resident. Suppose that $\sum_{i=1}^{n} a_i^2 = n^2 - n$. Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbours are friends. Determine all possible values of k.

2. Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

3. There is a finite number of points in the plane, such that every triangle has area at most 1. Prove that the points can be covered by a) a rectangle b) a triangle of area 4

4. Show that a simple graph with n vertices ($n \geq 3$) is Hamiltonian if each vertex has degree $n/2$ or greater.

5. Given n red points and n blue points in the plane in general position (no 3 of them are aligned), prove that there exists a pairing of the red points with the blue points such that the segments it draws are all disjoint.

6. What is the maximal number of a) bishops b) rooks c) knights such that no two of them can hit each other?

7. An alien race has three genders: male, female, and emale. A married triple consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual. The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper. Show that if n is even and $k = n/2$, then it might be impossible to create even one married triple. Show that if $k \geq 3n/4$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.