# Problems - 2013.11.07

P. Maga & P. P. Pach

1. Let $R$ be the region consisting of the points $(x, y)$ of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region $R$ and find its area.

2. On the $xy$-plane let $O = (0, 0)$, $A = (0, 1)$ and for $i \in \mathbb{N}$, let $B_i = (i, 1)$. Denote by $\alpha_i$ the angle $\angle AB_iO$.

   (a) $\alpha_1 + \alpha_2 + \alpha_3 =?$

   (b) $\sum_{i=1}^{\infty} \alpha_i =?$

3. Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2 + n + 1)$, where Arccot $t$ for $t \geq 0$ denotes the number $\theta$ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

4. Let $T$ be an acute triangle. Inscribe a rectangle $R$ in $T$ with one side along a side of $T$ . Then inscribe a rectangle $S$ in the triangle formed by the side of $R$ opposite the side on the boundary of $T$, and the other two sides of $T$, with one side along the side of $R$. For any polygon $X$ let $A(X)$ denote the area of $X$. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where $T$ ranges over all triangles and $R, S$ over all rectangles as above.

5. Let $C$ be the unit circle $x^2 + y^2 = 1$. A point $p$ is chosen randomly on the circumference $C$ and another point $q$ is chosen randomly from the interior of $C$ (these points are chosen independently and uniformly over their domains). Let $R$ be the rectangle with sides parallel to the $x$ and $y$-axes with diagonal $pq$. What is the probability that no point of $R$ lies outside of $C$?

6. Inscribe a rectangle of base $b$ and height $h$ in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of $h$ do the rectangle and triangle have the same area?

7. Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to $5/2$.

8. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.