Problems - 2013.10.24

P. Maga & P. P. Pach

0. Let $f : \mathbb{F}^2 \to \mathbb{F}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all $x, y, z \in \mathbb{F}$. Prove that there is a function $g : \mathbb{F} \to \mathbb{F}$ and a constant $c \in \mathbb{F}$ such that $f(x, y) = g(x) - g(y) + c$ for all $x, y \in \mathbb{F}$.

1. Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

2. Start with a finite sequence $a_1, a_2, \dots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

3. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

4. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \dots, h(p^2 - 1)$ are distinct modulo p^2. Show that $h(0), h(1), \dots, h(p^3 - 1)$ are distinct modulo p^3.

5. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that
$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$
and $\deg P > \deg Q$.

6. Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

Hard nuts

7. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \dots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

8. Let
$$\begin{aligned} f(z) &= az^4 + bz^3 + cz^2 + dz + e \\ &= a(z - r_1)(z - r_2)(z - r_3)(z - r_4) \end{aligned}$$
where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.