Problems - 2013.10.17

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1. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers
$$c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \ldots, c_n r^n + c_{n-1} r_{n-1} + \ldots + c_1 r$$
are integers.

2. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

3. Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

4. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

5. You have coins $C_1, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of fallings heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answers as a rational function of n.

Hard nuts

6. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions
$$f' = 2f^2gh + \frac{1}{gh}, \quad f(0) = 1;$$
$$g' = fg^2h + \frac{4}{fh}, \quad g(0) = 1;$$
$$h' = 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.$$
Find an explicit formula for $f(x)$, valid in some open interval around 0.

7. Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

8. Show that the improper integral
$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)dx$$
converges.