Problems - 2013.10.10

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1. Let $p_1, \dots, p_k, q_1, \dots, q_l$ be prime numbers. Prove that there is a prime number P such that $p_1, \dots, p_k$ are all quadratic residues and $q_1, \dots, q_l$ are all quadratic nonresidues modulo P.

2. For a given prime p, denote by $l(p)$ the least quadratic nonresidue. Prove that for any $\varepsilon > 0$, $l(p) < \varepsilon p$ if p is large enough.

3. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \qquad n \geq 1.$$

4. Let t be a fixed integer. Show that for any prime number p, one can find a positive integer n such that

$$(3 - 7t)2^n + (18t - 9)3^n + (6 - 10t)4^n$$

is divisible by p.

5. Determine all positive integers $n \geq 2$ that satisfy the following condition: for all a and b relatively prime to n we have $a \equiv b \pmod{n}$ if and only if $ab \equiv 1 \pmod{n}$.

6. Let p be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$

7. For which prime p does there exist a cubic polynomial f of integer coefficients such that the leading coefficient is not divisible by p and $f(1), \dots, f(p)$ are distinct modulo p?