Problems - 2013.10.03

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1. The PUTNAM awards banquet at the U.S. Department of State has assigned seating. However, after sitting down, the table #1 group finds that by mistake, they have all sat down at the wrong seats (although their correct seat is somewhere else at that table). Show that it is possible for them to simultaneously rotate around the table by some distance, such that at least two of them end up in the correct place.

2. Given n integers, prove that some nonempty subset of them has sum divisible by n.

3. Let A be a set of $n+1$ integers from $\{1, 2, \ldots, 2n\}$. Prove that some element of A divides another.

4. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

5. Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that
$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

6. Prove that every sequence of n^2 distinct numbers contains a subsequence of length n which is monotone (i.e. either always increasing or always decreasing).

7. A flea-beetle is jumping in every second with the same unknown vector starting from the origin. In every second we try to crush it with a bomb of a fixed range. For which n can we surely crush the flea-beetle in $\mathbb{R}^n$?

Hard nuts

8. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

9. Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.