Problems - 2013.09.23

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1. Given a finite simple graph on at least two vertices. Prove that there are two vertices of the same degree.

2. Given a finite simple graph. Prove that if it is not connected, then its complement is connected.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y and z. Prove that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

4. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

5. Given a positive integer n, what is the largest k such that the numbers $1, \dots, n$ can be put into k boxes such that the sum in each box is the same?

6. Let n be an even positive integer. Write the positive numbers $1, \dots, n^2$ in an $n \times n$ grid $(a_{i,j})_{1 \le i,j \le n}$ such that $a_{i,j} = (i - 1)n + j$. Color the grid such that in each row and in each column, half of the squares are red, and the other half are black. Prove that in any such coloring, the sum of the red numbers equals the sum of the black numbers.

7. Let a_j, b_j, c_j be integers for $1 \le j \le N$. Assume that for all j, at least one of them is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j.

Hard nuts

8. Given a finite simple graph. At each vertex, there is a lamp (of two states, on and off) and a switch. Each change in a switch changes the state of the corresponding and the neighboring lamps. Initially all lamps are off. Show that there is a sequence of changes that makes all lamps on.

9. Given a finite simple graph, a perfect matching is a matching which matches all vertices of the graph. Prove that if the degree of each vertex is even, then the number of perfect matches is even.