Problems - 2013.04.30

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1. Show that for any positive integer n, there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix, and that all such matrices have positive determinant.

2. Suppose that A, B are real matices such that $A^2 + B^2 = AB$. Prove that if $BA - AB$ is invertible, then n is divisible by 3.

3. Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2013}. ($\lfloor a \rfloor$ means the largest integer $\leq a$.)

4. Let $1, 2, 3, \ldots, 2012, 2013, 2014, 2016, 2019, 2023, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2013$ and $x_{k+1} = x_k + x_{k-2012}$ for $k \geq 2013$. Show that the sequence has 2012 consecutive terms each divisible by 2013.