Problems - 2013.04.23

P. Maga & P. P. Pach

1. For which values of n, $\sum_{k=1}^{n} 1/k$ is an integer?

2. Given 4 points on the plane, each lying on a side of an invisible square. Construct such a square.

3. Given a graph on n vertices, denote by m the lowest degree. (a) Prove that if $m \geq 3$, then the graph contains an even cycle. (b) Show (by an example for each n) that there exists a graph without an odd cycle, satisfying further $m \geq (n-1)/2$. (c) Prove that if $m \geq (n+1)/2$, then the graph contains an odd cycle.

4. A graph is said to be *almost planar*, if it can be realized in the plane such that each edge intersects at most one another. Prove there are positive constants c, c^* satisfying that for each $n \geq 2$, (a) there exists an almost planar graph with n vertices and at least cn edges; (b) every almost planar graph with n vertices has at most $c^* n$ edges.

5. Let $S \subseteq \mathbb{R}$ be a set of positive Lebesgue measure. Prove its cardinality is continuum.

6. Determine the continuously differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ that satisfy $f^2 + g^2 = f'^2 + g'^2 = 1$.

7. Let N be a positive integer. A set $S \subseteq \{1, \dots N\}$ is said to be *admissible* if there are no distinct elements $a, b, c \in S$ with $a|b|c$. At most how many elements can be in S?

Hard nuts

8. Assume f is a bounded, continuous function on the strip $0 \leq \Im z \leq 1$, which is holomorphic in the interior. For $0 \leq x \leq 1$, set $N(x) = \sup_{y \in \mathbb{R}} |f(x+iy)|$. Prove that for each $0 \leq x \leq 1$, $N(x) \leq \max(N(0), N(1))$, moreover $N(x) \leq N(0)^{1-x} N(1)^x$.

9. Define the sequence $(a_n)_{n \in \mathbb{N}}$ as $a_0 = 2$, $a_1 = 4$,

$$a_n = \frac{a_{n-1}a_{n-2}}{2} + a_{n-1} + a_{n-2}, \qquad n \geq 2.$$

Determine all prime numbers p satisfying $p|(a_m - 1)$ for a suitable m.