Problems - 2013.04.09

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1. Prove that a group is finite if and only if it has finitely many subgroups.

2. Let G be a group for which $Aut(G)$ is trivial. What can be G?

3. Let F_2 be the free group generated by the elements x, y. Let N be its normal subgroup generated by x^2 and y^2. Prove that $G = F_2/N$ is infinite.

4. Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$. Does the same conclusion hold if a and b are elements of an arbitrary non-commutative group?

5. Denote by S_n the symmetric group on n elements and by 1 its identity. Prove that if $n = 3$ or $n = 5$, then for any $1 \neq \pi_1 \in S_n$, there exists $\pi_2 \in S_n$ such that $S_n = \langle \pi_1, \pi_2 \rangle$. Prove that this does not hold for $n = 4$.

6. For an arbitrary set X, denote by S_X the group of $X \to X$ bijections. Prove that each element of S_X can be written as the product of two involutions ($a \in S_X$ is an involution, if $a^2 = \mathrm{id}_X$).

7.* Assume R is a ring with unit element 1. Prove that if $a, b \in R$ satisfy that $1 - ab$ is invertible, then $1 - ba$ is also invertible.

Hard nuts

8. Let R be a ring of characteristic zero (not necessarily commutative). Let e, f and g be idempotent elements of R satisfying $e + f + g = 0$. Show that $e = f = g = 0$. (R is of characteristic zero means that, if $a \in R$ and n is a positive integer, then $na \neq 0$ unless $a = 0$. An idempotent x is an element satisfying $x = x^2$.)

9. Determine the infinite abelian groups with the property that all their proper subgroups are finite. (Warm-up: does there exist any such group?)

10. Let G be a finite group. For arbitrary sets $U, V, W \subset G$, denote by N_{UVW} the number of triples $(x, y, z) \in U \times V \times W$ for which xyz is the unity. Suppose that G is partitioned into three sets A, B and C (i.e sets A, B, C are pairwise disjoint and $G = A \cup B \cup C$). Prove that $N_{ABC} = N_{CBA}$.

11. Denote by S_n the group of permutations of the sequence $\{1, 2, \ldots, n\}$. Suppose that G is a subgroup of S_n such that for every $\pi \in G \setminus \{e\}$ there exists a unique $k \in \{1, 2, \ldots, n\}$ for which $\pi(k) = k$. (Here e is the unit element in the group S_n.) Show that this k is the same for all $\pi \in G \setminus \{e\}$.